MUSIC: A MATHEMATICAL OFFERING

Since the time of the Ancient Greeks, much has been written about the relation between mathematics and music: from harmony and number theory to musical patterns and group theory. Benson provides a wealth of information here to enable the teacher, the student or the interested amateur to understand, at varying levels of technicality, the real interplay between these two ancient disciplines.

The story is long as well as broad and involves physics, biology, psychoacoustics, the history of science and digital technology as well as, of course, mathematics and music. Fundamental to it is how we actually hear sound, so the book starts with the structure of the human ear and its relationship with Fourier analysis. Combining this with the mathematics of musical instruments leads to the ideas of consonance and dissonance, and from there to an understanding of the development of scales and temperaments. Later chapters introduce some separate but related threads involving symmetry in music and the modern introduction of digital techniques to produce and analyze music and sound. This is a must-have book if you want to know about the music of the spheres or digital music, and many things in between.

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To Christine Natasha

Ode to an Old Fiddle from *The Musical World* (London, 1834), quoted in Nicolas Slonimsky's *Book of Musical Anecdotes*, Schirmer, 1998.

THE POOR FIDDLER'S ODE TO HIS OLD FIDDLE

Torn
Worn
Oppressed I mourn
B a d
S a d
Three-quarters mad
Money gone
Credit none
Duns at door
Half a score
Wife in lain
Twins again
Others ailing
Nurse a railing
Billy hooping
Betsy crouping
Besides poor Joe
With fester'd toe.
Come, then, my Fiddle,
Come, my time-worn friend,
With gay and brilliant sounds
Some sweet tho' transient solace lend,
Thy polished neck in close embrace
I clasp, whilst joy illumines my face.
When o'er thy strings I draw my bow,
My drooping spirit pants to rise;
A lively strain I touch—and, lo!
I seem to mount above the skies.
There on Fancy's wing I soar
Heedless of the duns at door;
Oblivious all, I feel my woes no more;
But skip o'er the strings,
As my old Fiddle sings,
"Cheerily oh! merrily go!
"PRESTO! good master,
"You very well know
"I will find Music,
"If you will find bow,
"From E, up in alto, to G, down below."
Fatigued, I pause to change the time
For some *Adagio*, solemn and sublime.
With graceful action moves the sinuous arm;
My heart, responsive to the soothing charm,
Throbs equably; whilst every health-corroding care
Lies prostrate, vanquished by the soft mellifluous air.
More and more plaintive grown, my eyes with tears o'erflow,
And Resignation mild soon smooths my wrinkled brow.
Reedy Hautboy may squeak, wailing Flauto may squall,
The Serpent may grunt, and the Trombone may bawl;
But, by Poll,* my old Fiddle's the prince of them all.
Could e'en Dryden return, thy praise to rehearse,
His Ode to Cecilia would seem rugged verse.
Now to thy case, in flannel warm to lie,
Till call'd again to pipe thy master's eye.
*Apollo.

Contents

Preface

This book has been a long time in the making. My interest in the connections between mathematics and music started in earnest in the early nineties, when I bought a second-hand synthesizer. This beast used a simple frequency modulation model to produce its sounds, and I was fascinated at how interesting and seemingly complex the results were. Trying to understand what was going on led me on a long journey through the nature of sound and music and its relations with mathematics, a journey that soon outgrew these origins.

Eventually, I had so much material that I decided it would be fun to try to teach a course on the subject. This ran twice as an undergraduate mathematics course in 2000 and 2001, and then again in 2003 as a Freshman Seminar. The responses of the students were interesting: each seemed to latch onto certain aspects of the subject and find others less interesting; but which parts were interesting varied radically from student to student.

With this in mind, I have tried to put together this book in such a way that different sections can be read more or less independently. Nevertheless, there *is* a thread of argument running through the book; it is described in the introduction. I strongly recommend the reader not to try to read this book sequentially, but at least to read the introduction first for orientation before dipping in.

The mathematical level of different parts of the book varies tremendously. So if you find some parts too taxing, don't despair. Just skip around a bit.

I've also tried to write the book in such a way that it can be used as the text for an undergraduate course. So there are exercises of varying difficulty, and outlines of answers in an appendix in the online version.

Cambridge University Press has kindly allowed me to keep a version of this book available for free online at www.maths.abdn.ac.uk/~bensondj/html/maths-music.html. No version of the online book will ever be identical to the printed

book. Some ephemeral information is contained in the online version that would be inappropriate for the printed version; and the quality of the images in the printed version is much higher than in the online version. Moreover, the online version is likely to continue to evolve, so that *references to it will always be unstable*.

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