

MUSIC: A MATHEMATICAL OFFERING

Since the time of the Ancient Greeks, much has been written about the relation between mathematics and music: from harmony and number theory to musical patterns and group theory. Benson provides a wealth of information here to enable the teacher, the student or the interested amateur to understand, at varying levels of technicality, the real interplay between these two ancient disciplines.

The story is long as well as broad and involves physics, biology, psychoacoustics, the history of science and digital technology as well as, of course, mathematics and music. Fundamental to it is how we actually hear sound, so the book starts with the structure of the human ear and its relationship with Fourier analysis. Combining this with the mathematics of musical instruments leads to the ideas of consonance and dissonance, and from there to an understanding of the development of scales and temperaments. Later chapters introduce some separate but related threads involving symmetry in music and the modern introduction of digital techniques to produce and analyze music and sound. This is a must-have book if you want to know about the music of the spheres or digital music, and many things in between.

DAVE BENSON is Sixth Century Professor of Pure Mathematics at the University of Aberdeen. He has held positions in Georgia, Oxford, and at Northwestern and Yale, and visiting positions at many places throughout the world. He is a keen amateur singer and has performed in many operas.

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To Christine Natasha

Ode to an Old Fiddle from *The Musical World* (London, 1834), quoted in Nicolas Slonimsky’s *Book of Musical Anecdotes*, Schirmer, 1998.

THE POOR FIDDLER’S ODE TO HIS OLD FIDDLE

Torn
Worn
Oppressed I mourn
 B a d
 S a d
Three-quarters mad
 Money gone
 Credit none
 Duns at door
 Half a score
 Wife in lair
 Twins again
 Others ailing
 Nurse a railing
 Billy hooping
 Betsy crouching
 Besides poor Joe
 With fester’d toe.
Come, then, my Fiddle,
Come, my time-worn friend,
With gay and brilliant sounds
Some sweet tho’ transient solace lend,
Thy polished neck in close embrace
I clasp, whilst joy illumines my face.
When o’er thy strings I draw my bow,
My drooping spirit pants to rise;
A lively strain I touch—and, lo!
I seem to mount above the skies.
There on Fancy’s wing I soar
Heedless of the duns at door;
Oblivious all, I feel my woes no more;
But skip o’er the strings,
As my old Fiddle sings,
“Cheerily oh! merrily go!
“PRESTO! good master,
“You very well know
“I will find Music,
“If you will find bow,
“From E, up in alto, to G, down below.”
Fatigued, I pause to change the time
For some *Adagio*, solemn and sublime.
With graceful action moves the sinuous arm;
My heart, responsive to the soothing charm,
Throbs equably; whilst every health-corroding care
Lies prostrate, vanquished by the soft mellifluous air.
More and more plaintive grown, my eyes with tears o’erflow,
And Resignation mild soon smooths my wrinkled brow.
Reedy Hautboy may squeak, wailing Flauto may squall,
The Serpent may grunt, and the Trombone may bawl;
But, by Poll,* my old Fiddle’s the prince of them all.
Could e’en Dryden return, thy praise to rehearse,
His Ode to Cecilia would seem rugged verse.
Now to thy case, in flannel warm to lie,
Till call’d again to pipe thy master’s eye.
*Apollo.

Contents

<i>Preface</i>	<i>page</i> xi
<i>Acknowledgements</i>	xiii
Introduction	1
1 Waves and harmonics	5
1.1 What is sound?	5
1.2 The human ear	7
1.3 Limitations of the ear	13
1.4 Why sine waves?	17
1.5 Harmonic motion	18
1.6 Vibrating strings	19
1.7 Sine waves and frequency spectrum	21
1.8 Trigonometric identities and beats	23
1.9 Superposition	26
1.10 Damped harmonic motion	28
1.11 Resonance	31
2 Fourier theory	36
2.1 Introduction	37
2.2 Fourier coefficients	38
2.3 Even and odd functions	44
2.4 Conditions for convergence	46
2.5 The Gibbs phenomenon	50
2.6 Complex coefficients	54
2.7 Proof of Fejér’s theorem	55
2.8 Bessel functions	58
2.9 Properties of Bessel functions	61
2.10 Bessel’s equation and power series	63
2.11 Fourier series for FM feedback and planetary motion	68
2.12 Pulse streams	71

viii	<i>Contents</i>	
2.13	The Fourier transform	73
2.14	Proof of the inversion formula	77
2.15	Spectrum	80
2.16	The Poisson summation formula	81
2.17	The Dirac delta function	82
2.18	Convolution	86
2.19	Cepstrum	88
2.20	The Hilbert transform and instantaneous frequency	89
3	A mathematician's guide to the orchestra	91
3.1	Introduction	91
3.2	The wave equation for strings	92
3.3	Initial conditions	100
3.4	The bowed string	103
3.5	Wind instruments	107
3.6	The drum	112
3.7	Eigenvalues of the Laplace operator	117
3.8	The horn	120
3.9	Xylophones and tubular bells	122
3.10	The mbira	130
3.11	The gong	133
3.12	The bell	138
3.13	Acoustics	142
4	Consonance and dissonance	144
4.1	Harmonics	144
4.2	Simple integer ratios	145
4.3	History of consonance and dissonance	148
4.4	Critical bandwidth	151
4.5	Complex tones	152
4.6	Artificial spectra	153
4.7	Combination tones	155
4.8	Musical paradoxes	158
5	Scales and temperaments: the fivefold way	161
5.1	Introduction	162
5.2	Pythagorean scale	162
5.3	The cycle of fifths	164
5.4	Cents	165
5.5	Just intonation	167
5.6	Major and minor	168
5.7	The dominant seventh	170
5.8	Commas and schismas	171

	<i>Contents</i>	ix
5.9	Eitz’s notation	172
5.10	Examples of just scales	174
5.11	Classical harmony	181
5.12	Meantone scale	185
5.13	Irregular temperaments	189
5.14	Equal temperament	198
5.15	Historical remarks	202
6	More scales and temperaments	210
6.1	Harry Partch’s 43 tone and other just scales	210
6.2	Continued fractions	214
6.3	Fifty-three tone scale	223
6.4	Other equal tempered scales	227
6.5	Thirty-one tone scale	228
6.6	The scales of Wendy Carlos	231
6.7	The Bohlen–Pierce scale	233
6.8	Unison vectors and periodicity blocks	237
6.9	Septimal harmony	242
7	Digital music	245
7.1	Digital signals	245
7.2	Dithering	247
7.3	WAV and MP3 files	248
7.4	MIDI	251
7.5	Delta functions and sampling	251
7.6	Nyquist’s theorem	254
7.7	The z -transform	256
7.8	Digital filters	257
7.9	The discrete Fourier transform	261
7.10	The fast Fourier transform	263
8	Synthesis	265
8.1	Introduction	265
8.2	Envelopes and LFOs	266
8.3	Additive synthesis	268
8.4	Physical modelling	270
8.5	The Karplus–Strong algorithm	273
8.6	Filter analysis for the Karplus–Strong algorithm	275
8.7	Amplitude and frequency modulation	276
8.8	The Yamaha DX7 and FM synthesis	280
8.9	Feedback, or self-modulation	287
8.10	CSound	291
8.11	FM synthesis using CSound	298

x	<i>Contents</i>	
8.12	Simple FM instruments	300
8.13	Further techniques in CSound	304
8.14	Other methods of synthesis	308
8.15	The phase vocoder	309
8.16	Chebyshev polynomials	309
9	Symmetry in music	312
9.1	Symmetries	312
9.2	The harp of the Nzakara	322
9.3	Sets and groups	324
9.4	Change ringing	329
9.5	Cayley’s theorem	331
9.6	Clock arithmetic and octave equivalence	333
9.7	Generators	335
9.8	Tone rows	337
9.9	Cartesian products	339
9.10	Dihedral groups	340
9.11	Orbits and cosets	342
9.12	Normal subgroups and quotients	343
9.13	Burnside’s lemma	345
9.14	Pitch class sets	348
9.15	Pólya’s enumeration theorem	353
9.16	The Mathieu group M_{12}	358
Appendix A	Bessel functions	361
Appendix B	Equal tempered scales	365
Appendix C	Frequency and MIDI chart	367
Appendix D	Intervals	368
Appendix E	Just, equal and meantone scales compared	372
Appendix F	Music theory	374
Appendix G	Recordings	381
	<i>References</i>	386
	<i>Bibliography</i>	389
	<i>Index</i>	393

Preface

This book has been a long time in the making. My interest in the connections between mathematics and music started in earnest in the early nineties, when I bought a second-hand synthesizer. This beast used a simple frequency modulation model to produce its sounds, and I was fascinated at how interesting and seemingly complex the results were. Trying to understand what was going on led me on a long journey through the nature of sound and music and its relations with mathematics, a journey that soon outgrew these origins.

Eventually, I had so much material that I decided it would be fun to try to teach a course on the subject. This ran twice as an undergraduate mathematics course in 2000 and 2001, and then again in 2003 as a Freshman Seminar. The responses of the students were interesting: each seemed to latch onto certain aspects of the subject and find others less interesting; but which parts were interesting varied radically from student to student.

With this in mind, I have tried to put together this book in such a way that different sections can be read more or less independently. Nevertheless, there *is* a thread of argument running through the book; it is described in the introduction. I strongly recommend the reader not to try to read this book sequentially, but at least to read the introduction first for orientation before dipping in.

The mathematical level of different parts of the book varies tremendously. So if you find some parts too taxing, don't despair. Just skip around a bit.

I've also tried to write the book in such a way that it can be used as the text for an undergraduate course. So there are exercises of varying difficulty, and outlines of answers in an appendix in the online version.

Cambridge University Press has kindly allowed me to keep a version of this book available for free online at www.maths.abdn.ac.uk/~bensondj/html/maths-music.html. No version of the online book will ever be identical to the printed

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book. Some ephemeral information is contained in the online version that would be inappropriate for the printed version; and the quality of the images in the printed version is much higher than in the online version. Moreover, the online version is likely to continue to evolve, so that *references to it will always be unstable*.

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