1

Classical theory of radiation

Although most of the electromagnetic radiation from many natural and laboratory plasmas is atomic in origin and therefore subject to quantum effects, it remains useful to introduce some of the basic radiative processes via classical theory. Other important foundations of plasma spectroscopy are atomic physics and plasma physics, especially the statistical mechanics of ionized gases. Generally, the basic theory is well established in these parent disciplines. However, the large variety of processes contributing to emission or absorption spectra often requires more or less drastic simplifications in their theoretical description or computer modeling. Critical experiments are playing an essential role in checking the reliability of various models and in delineating the region of their applicability.

Plasma spectroscopy, although being a highly specialized subfield, is at the same time a very interdisciplinary science. It not only owes its origins largely to astronomy, but also returns to astronomy and astrophysics methods of analysis of spectra and a multitude of basic data, which have both been subjected to experimental scrutiny. The state of stellar plasmas is significantly influenced by radiation, and the latter is more or less controlled by radiative energy transfer. Internally consistent treatments of the states of matter and radiation first developed by astronomers are now also becoming important for the description of plasma experiments.

This give-and-take between the various sciences and scientists, experimentalists, theoreticians, computer modelers, and observers, is responsible for much of the progress made in our understanding and working knowledge of plasma radiation physics in the three decades since the first monograph on plasma spectroscopy (Griem 1964), an early review (Cooper 1966) and an introductory text (Marr 1968) were written. This progress is not only reflected in astrophysical data tables (Allen 1973,
1 Classical theory of radiation

Lang 1980), but also in a number of other monographs covering the physics of atoms and ions in plasmas (Sobelman, Vainshtein and Yukov 1981, Janev, Presnyakov and Shevelko 1985, Shevelko and Vainshtein 1993, Lisitsa 1994, Kobzev, Iakubov and Popovich 1995). References to tables of spectroscopic data can be found in the introduction to chapter 3.

1.1 Electromagnetic equations and fields from moving charges

The International System (S.I.) of units will be generally used here, although most results will be expressed in terms of fundamental constants such that evaluation, e.g., in cgs units will be straightforward. To discuss experimental results, wavelengths will usually be in Angstrom units and densities per cm$^3$.

Maxwell’s equations lead to the inhomogeneous wave equations for vector and scalar potentials, namely,

$$\frac{1}{c^2} \ddot{\mathbf{A}} - \nabla^2 \mathbf{A} = \mu_0 \rho \mathbf{v}$$  \hspace{1cm} (1.1)

and

$$\frac{1}{c^2} \ddot{\phi} - \nabla^2 \phi = \frac{1}{\varepsilon_0} \rho,$$  \hspace{1cm} (1.2)

if the Lorentz condition

$$\nabla \cdot \mathbf{A} + \varepsilon_0 \mu_0 \dot{\phi} = 0$$  \hspace{1cm} (1.3)

is also imposed. Here the vacuum values of dielectric constant and magnetic permeability are $\varepsilon_0 = (4\pi \times 9 \times 10^9)^{-1} = 8.85 \times 10^{-12} \text{C}^2\text{N}^{-1}\text{m}^{-2}$ and $\mu_0 = 4\pi \times 10^{-7} \text{W}\text{A}^{-1}\text{m}^{-1}$, and $\rho$ and $\mathbf{v}$ are charge densities and velocities. Also, the units are coulomb (C) for charge, newton (N) for force and weber (W) for magnetic flux (1 W = 1 tesla m$^2$), while the current is in amperes (A), not to be confused with the vector potential.

A special solution of the inhomogeneous wave equations is given by the Lienard-Wiechert potentials (Jackson 1962, Landau and Lifshitz 1951), which account for retardation effects and assume point charges as sources of the field. The electric and magnetic fields then follow from

$$\mathbf{E} = -\nabla \phi - \dot{\mathbf{A}}$$  \hspace{1cm} (1.4)

and

$$\mathbf{B} = \nabla \times \mathbf{A},$$  \hspace{1cm} (1.5)

and in our case we have to add the general solution of the homogeneous wave equation for $\mathbf{A}$. As shown, e.g., in Heitler (1954), a moving electron
1.2 Emission of radiation

produces fields according to the above equations given by

\[
E = -\frac{e}{4\pi \varepsilon_0} \left\{ \left( 1 - \frac{v^2}{c^2} \right) \left( r + \frac{v}{c} \right) \right. \\
- \frac{1}{c^2} \hat{r} \times \left[s \left( r + \frac{v}{c} \right) \times \hat{v} \right] \left( r + \frac{1}{c} v \cdot \hat{r} \right)^{-3}, \tag{1.6}
\]

\[
B = \frac{1}{rc} E \times r, \quad \tag{1.7}
\]

where \( r \) is the vector from the field point to the charge. The magnetic field is always transverse. The electric field becomes purely transverse only at large distances, i.e., in the wave zone, where the first term in (1.6) is negligible. In the wave zone, the fields are therefore proportional to the acceleration.

1.2 Emission of radiation

Since Maxwell’s equations are linear, fields from a number of closely spaced point charges \( e_k \) at positions \( r_k = \vec{r} + \vec{x}_k \) can be obtained by superposition of fields according to (1.6) and (1.7). Assuming the average distance \( |\vec{r}| \) from the field point to the charge cluster to be much larger than the \( |\vec{x}_k| \), and \( v/c \) to be small, the far field then becomes

\[
E = \frac{1}{4\pi \varepsilon_0} \vec{r} \times \vec{r} \times \sum_k e_k \vec{x}_k / c^2 |\vec{r}|^3, \tag{1.8}
\]

\[
B = \frac{1}{4\pi \varepsilon_0} \vec{r} \times \sum_k e_k \vec{x}_k / c^3 |\vec{r}|^2. \tag{1.9}
\]

The dominant contributions to the far field are seen to be proportional to the second derivative of the electric dipole moment of the charge cluster. If retardation effects are accounted for more accurately and all first order terms in \( v/c \) are included, additional electric quadrupole terms arise in (1.8) and (1.9) which are usually only important if the dipole moment vanishes. Furthermore, if magnetic dipoles are considered as sources, corresponding to small current loops on the right-hand side of (1.1), expressions analogous to (1.8) and (1.9) are obtained with \( \mathbf{E} \) and \( \mathbf{B} \) exchanged and the magnetic dipole moment replacing the electric dipole moment (Jackson 1962). For atomic electrons, this magnetic dipole moment is typically smaller than electric dipole moments by a factor \( \sim az/n^2 \), where \( z \) is the effective nuclear charge and \( n \) the principal quantum number for the active electron (Cowan 1981, Sobel'man 1992). The first factor is the fine-structure constant, \( \alpha = e^2 / 4\pi \varepsilon_0 \hbar c \approx 1/137. \)
1 Classical theory of radiation

To calculate the radiated power, we need the Poynting vector,

\[ S = E \times H = -\frac{1}{(4\pi)^2 \varepsilon_0} |\mathbf{r}| \sum_k e_k |\mathbf{x}_k|^2 |\mathbf{r}| / c^3 |\mathbf{r}|^5 \]

\[ = -\frac{1}{(4\pi)^2 \varepsilon_0} |\mathbf{r}| \sum_k e_k |\mathbf{x}_k|^2 |\mathbf{r}| \sin^2 \theta / c^3 |\mathbf{r}|^3. \]  

(1.10)

The minus sign appears because \( \mathbf{r} \) is from the field point to the center of the charge distribution, and \( \theta \) in the second version is the angle between \( \mathbf{r} \) and the second derivative of the dipole moment. Frequently one is interested only in the total radiated power, which follows by integrating \( S \), e.g., over a large spherical surface surrounding the charge distribution. This power is

\[ P_e = \frac{1}{6\pi \varepsilon_0 c^3} |\sum_k e_k |\mathbf{x}_k|^2 \]

\[ = \frac{2 m r_0}{3 c} |\sum_k |\mathbf{x}_k|^2. \]  

(1.11)

In the second version, which only holds for electrons, the classical electron radius \( r_0 = e^2 / 4\pi \varepsilon_0 mc^2 \approx 2.818 \times 10^{-15} \text{m} \) is used to simplify the expression.

Finally, for a harmonic oscillator, i.e., \( \sum e_k x_k = ex(t) = ex_0 \cos \omega_0 t \), etc., the time-averaged power is

\[ \bar{P}_e = \frac{1}{3c} mr_0 \omega_0^4 |x_0|^2. \]  

(1.12)

This is the quantity of primary interest in quantitative spectroscopy.

Another case of great interest involves radiation from free electrons in magnetized laboratory and natural plasmas (Bekefi 1966). The accelerated motion in this case causes cyclotron, cyclotron-harmonic, and synchrotron radiation, in the order of increasing electron energy toward relativistic energies. To describe this radiation, one must return to (1.6) and (1.7), inserting the helical motion in a magnetic field. The interested reader is referred to Bekefi’s (1966) book.

1.3 Absorption by harmonic oscillators

If a harmonic oscillator is initially not excited but, say, at \( t = 0 \) exposed to incident electromagnetic waves, it will be driven into oscillations. Interaction with waves at \( t > 0 \) will then usually lead to an increase of the oscillator’s energy, i.e., to absorption or loss of wave energy. An appropriate equation of motion for the oscillator is

\[ \ddot{x} + \omega_0^2 x = \frac{e}{m} \sum E_\omega \cos(\omega t + \delta_\omega). \]  

(1.13)
1.3 Absorption by harmonic oscillators

Here the $E_\omega$ are the wave amplitudes of the initial waves and the $\delta_\omega$ their phases, and the sum is over frequencies, modes, and phases. The special solution required here is

$$x(t) = \frac{e^2}{m} \sum \frac{E_\omega}{\omega_0^2 - \omega^2} \left[ \cos(\omega_0 t + \delta_\omega) - \cos(\omega_0 t + \delta_\omega) \right].$$  \hspace{1cm} (1.14)

Writing the wave field as $\sum' E'_\omega \cos(\omega t + \delta')$, the absorbed power or work done by the field is

$$dP_a = e \mathbf{x}(t) \cdot \sum' E'_\omega \cos(\omega t + \delta')$$

$$= \frac{e^2}{m} \sum E'^2_\omega \left[ -\frac{\omega}{\omega_0^2 - \omega^2} \sin(\omega t + \delta_\omega) + \frac{\omega_0}{\omega_0^2 - \omega^2} \sin(\omega_0 t + \delta_\omega) \right]$$

$$\times \cos(\omega t + \delta_\omega).$$ \hspace{1cm} (1.15)

(Nota: the free oscillations included in Griem 1964 need not be considered here.) On the average over many oscillations, the first term in the square-bracketed expression does not contribute. The average absorbed power is, therefore, using standard trigonometric formulas for the $\sin(\omega_0 t + \delta_\omega) \cos(\omega t + \delta_\omega)$ product and averaging over phases

$$d\bar{P}_a = \frac{e^2}{2m} \sum E'^2_\omega \frac{\omega_0}{\omega_0^2 - \omega^2} \frac{1}{\tau} \int_0^\tau \sin[(\omega_0 - \omega)t]dt$$

$$= \frac{e^2}{2m} \sum E'^2_\omega \frac{\omega_0}{\omega_0 + \omega} \frac{1 - \cos[(\omega_0 - \omega)\tau]}{\tau(\omega_0 - \omega)^2}.$$ \hspace{1cm} (1.16)

For large $\tau$ the last factor is proportional to the Dirac delta function $\delta(\omega - \omega_0)$. Replacing $\sum$ by $\int d\omega$ and using $\int_0^\infty dx(1 - \cos x)/x^2 = \pi$, the total absorbed power thus becomes

$$\bar{P}_a = \frac{\pi e^2}{4m} E^2_\omega.$$ \hspace{1cm} (1.17)

The spectral density $E^2_\omega$ of the waves' electric field is related to the spectral energy flux $\bar{S}(\omega)$, i.e., the Poynting vector magnitude per angular frequency interval, through

$$\bar{S}(\omega) = \frac{\mathbf{E}_\omega \times \mathbf{H}_\omega}{\mu_0 c}$$

$$= \frac{1}{\mu_0 c} E^2_\omega \cos^2(\omega t + \delta_\omega)$$

$$= c \frac{\varepsilon_0 E^2_\omega}{2},$$ \hspace{1cm} (1.18)

using (1.10) and $c^2 = 1/\varepsilon_0 \mu_0$. In terms of $\bar{S}(\omega)$, the absorbed power is finally

$$\bar{P}_a = \frac{\pi e^2}{2 \varepsilon_0 m c} \bar{S}(\omega).$$ \hspace{1cm} (1.19)
and we infer that the absorption cross section \( \sigma(\omega) \) is related to the first factor through

\[
\int \sigma_a(\omega) d\omega = \frac{\pi e^2}{2e_0 mc} = 2\pi^2 \rho_0 c. \tag{1.20}
\]

This classical formula agrees with the quantum mechanical result if the line absorption oscillator strength introduced in the next chapter is \( f = 1 \).

It is also noteworthy that the results obtained here depend on the random phases of the wave modes. One could, with different assumptions for the phases and initial conditions, even obtain a negative power, a classical analog of the induced emission process first postulated by Einstein (1917) for his derivation of Planck’s law.

### 1.4 Radiation damping

In our discussion of linear oscillators and their effects on electromagnetic waves, we so far implicitly assumed undamped oscillations of the oscillator. This is clearly inconsistent with the emission of radiation at a rate according to (1.12). To conserve energy, the energy of the oscillator, \( \frac{1}{2} m\omega_0^2 |x_0|^2 \), must decrease to make up for the radiated power. This suggests

\[
\frac{1}{2} m\omega_0^2 \frac{d}{dt} |x_0|^2 = -P_e = -\frac{1}{3c} m\rho_0 \omega_0^2 |x_0|^2 \tag{1.21}
\]

if the rate of change in the amplitude remains small enough not to invalidate (1.12).

The characteristic decay rate of the square of the amplitude is then estimated by

\[
\gamma = -\frac{1}{|x_0|^2} \frac{d}{dt} |x_0|^2 = \frac{2}{3c} \rho_0 \omega_0^2 \tag{1.22}
\]

provided \( \gamma \ll \omega_0 \) so that the time variation of \( x_0 \) is very slow.

Because the radiation fields now have an \( \exp[-i(\omega_0 - \frac{1}{2}\gamma)t] \) dependence on time rather than the \( \exp(-i\omega_0 t) \) dependence for constant amplitudes, their Fourier components \( \int_0^\infty dt \exp[i(\omega - \omega_0)t - \frac{1}{2}\gamma t] \) are given by \( \frac{1}{\gamma} - i(\omega - \omega_0)^{-1} \). The emitted power per frequency interval is proportional to the absolute value squared of these Fourier components, i.e.,

\[
\frac{dP_e}{d\omega} = \frac{1}{3c} m\rho_0 \omega_0^4 |x_0|^2 \frac{\gamma/2\pi}{(\gamma/2)^2 + (\omega - \omega_0)^2} \tag{1.23}
\]

if one uses a time average of \( |x_0|^2 \). Moreover, \( \gamma/2\pi \) is introduced as normalization factor so that the spectrally integrated power remains the same.
1.5 Scattering of radiation

The last factor in (1.23) is the normalized line shape function accounting for the damping of the oscillator by radiation. Collisions can cause a similar effect, if their duration is short. The line shape then remains the same (Lorentzian), but an effective collision frequency \( v_{\text{coll}} \) must be added to \( \gamma \). (Details concerning this collision or impact broadening and other spectral line broadening mechanisms are discussed in chapter 4.)

A more rigorous calculation of radiation damping involves the solution of the equation of motion of the harmonic oscillator in its own radiation field. The reaction of this field then results in damping of the oscillations and a very small frequency shift (Jackson 1962). These results agree with the above estimate only in the limit \( r_0/\lambda \approx r_0\omega_0/c \ll 1 \), for a given wavelength \( \lambda \). According to (1.22), this is equivalent to \( \gamma \ll \omega_0 \).

In the same regime, which covers all atomic radiation quite comfortably, one can describe the effects of radiation damping also by introducing an effective friction force into the equation of motion of the oscillator. The work done by this force \( F_r \) must correspond to the radiated energy, i.e.,

\[
- \int F_r \cdot \dot{x} dt = \frac{2}{3c} m r_0 \int |\ddot{x}|^2 dt, \tag{1.24}
\]

using (1.11). Integrating \( \dot{x} \cdot \ddot{x} \) by parts and averaging over an interval containing many periods gives

\[
F_r = \frac{2}{3c} m r_0 \ddot{x} \approx -\frac{2}{3c} m r_0 \omega_0^2 \ddot{x} = -\gamma m \ddot{x}. \tag{1.25}
\]

This is again as expected and shows that \( \gamma m \) is analogous to a coefficient of friction.

1.5 Scattering of radiation

The forced oscillations of a harmonic oscillator induced by the electric field of an electromagnetic field also cause emission of radiation. This is given by (1.12), if only the total scattered power is desired, or by the product of this expression and \( 3\sin^2\theta/8\pi \) if the differential scattered power per unit solid angle is required. The angle \( \theta \) is here between the direction of observation and the electric field vector of the primary wave, and \( 3/8\pi \) is needed to retain the total scattered power. Finally, the \( \sin^2\theta \) dependence is obtained from (1.10).

To calculate the quantity equivalent to \( x_0 \) in (1.12) one considers the equation of motion

\[
\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{e}{m} E_0 \exp(i\omega t) \tag{1.26}
\]

for the forced oscillator with radiation damping according to (1.25). The
stationary solution for $x(t)$ is

$$x = \frac{(e/m)E_0 \exp(i\omega t)}{\omega_0^2 - \omega^2 + i\gamma \omega}. \quad (1.27)$$

Substitution into (1.12), and again replacing the amplitude of the incident wave by the incident spectral energy flux according to (1.18), results in a scattered power

$$P_s = \frac{8\pi}{3} \frac{\omega^4}{r_0^2 (\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} S(\omega). \quad (1.28)$$

Here we omit the average sign over $S(\omega)$.

As in the case of absorption, the factor of $S(\omega)$ is a cross section, namely for scattering by a single oscillator

$$\sigma_s = \frac{8\pi}{3} \frac{\omega^4}{r_0^2 (\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}. \quad (1.29)$$

If an array of scatterers is involved, and if their positions are correlated, the different scattered waves will interfere. The result of such coherent scattering, therefore, depends critically on the relative phases of the various scattered waves.

For free electrons, that is $\omega_0 = 0$, we obtain the total Thomson scattering cross section

$$\sigma_{\text{Th}} = \frac{8\pi}{3} r_0^2. \quad (1.30)$$

For $\omega \approx \omega_0$ the approximation $\omega_0^2 - \omega^2 \approx 2\omega(\omega_0 - \omega)$ leads to the resonance scattering cross section

$$\sigma_{\text{res}} = \frac{2\pi}{3} \frac{r_0^2}{\omega_0^2 - \omega^2 + (\gamma/2)^2}. \quad (1.31)$$

Its frequency integral is $\int \sigma_{\text{res}} d\omega = 4\pi^2 r_0^2 \omega_0^2 / 3\gamma = 2\pi^2 r_0 c$, using (1.22). This integral is, therefore, the same as that of the absorption cross section given by (1.20). This suggests that in the present picture it makes no difference whether we speak of absorption followed by emission, or of resonant scattering or resonance fluorescence, at least if we average over directions and neglect collisions, etc.

Finally, if the angular dependence of the scattering is required, we must re-instate the factor $\sin^2 \theta$ discussed at the beginning of this section and then obtain with the additional normalization factor $8\pi / 3$ for the differential cross section

$$\frac{d\sigma_s}{d\Omega} = \frac{3}{8\pi} \sigma_s \sin^2 \theta = r_0^2 \frac{\omega^4 \sin^2 \theta}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}. \quad (1.32)$$

From this result those for $\omega_0 = 0$ and $\omega \approx \omega_0$ again follow easily.
1.6 Optical refractivity

In most natural and laboratory plasmas, densities are low enough that electromagnetic waves have phase velocities so close to $c$ that refraction of light rays and deviations from vacuum wavelengths can be ignored. (Actually, tabulated wavelengths for lines in and near the visible range are usually measured in air.) However, in very dense plasmas and also for wavelengths near strong lines, deviations of the refractive index from $n = 1$ can be important. This is frequently due to the polarization of free electrons but, especially in partially ionized gases, in some cases also due to bound electrons.

Excluding spectral regions very near resonance with lines, i.e., neglecting radiative and other damping, the dipole moments of the harmonic oscillators representing atoms, ions and electrons in a plasma are from (1.27)

$$e\mathbf{x} = \frac{(e^2/m)\mathbf{E}_0 \exp(i\omega t)}{\omega_0^2 - \omega^2}$$  \hspace{1cm} (1.33)

in a wave field of amplitude $\mathbf{E}_0$ and frequency $\omega$. For $N$ such dipoles per unit volume this corresponds to a volume polarization

$$\mathbf{P} = N e\mathbf{x} = N \frac{(e^2/m)\mathbf{E}_0 \exp(i\omega t)}{\omega_0^2 - \omega^2},$$  \hspace{1cm} (1.34)

neglecting the Lorentz-Lorenz correction for fields from the oscillators (Jackson 1962) in comparison to the original wave field. With these approximations the displacement vector is

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \equiv \varepsilon_0 n^2 \mathbf{E} = \varepsilon_0 \left[ 1 + \frac{(e^2/\varepsilon_0 m)N}{\omega_0^2 - \omega^2} \right] \mathbf{E}.$$  \hspace{1cm} (1.35)

It is consistent with the above approximations to write $n^2 - 1 = (n + 1)(n - 1) \approx 2(n - 1)$ so that we finally obtain

$$n = 1 + \frac{(e^2/2\varepsilon_0 m)N}{\omega_0^2 - \omega^2} = 1 + \frac{2\pi r_0 e^2 N}{\omega_0^2 - \omega^2}.$$  \hspace{1cm} (1.36)

An important special case is that of free electrons, i.e., $\omega_0 = 0$, giving

$$n = 1 - \frac{2\pi r_0 e^2 N_e}{\omega^2} = 1 - \frac{1}{2} \left( \frac{\omega_{pe}}{\omega^2} \right)^2 = 1 - \frac{r_0}{2 \pi} \frac{\omega_{pe}^2}{\omega^2} N_e.$$  \hspace{1cm} (1.37)

Here $\omega_{pe}$ is the electron plasma frequency. In plasmas, the phase velocity is therefore normally larger than $c$, unless contributions from bound electrons according to (1.36) dominate and we have $\omega < \omega_0$. (Remem-
ber, however, that this expression is not valid for $\omega \approx \omega_0$, see section 2.10.) Finally, in magnetized plasmas, the optical properties become anisotropic and depend on frequency in a much more complicated way (Bekefi 1966).