PART ONE

Poisson geometry and Morita equivalence

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Introduction

Poisson geometry is a "transitional" subject between noncommutative algebra and differential geometry (which could be seen as the study of a very special class of commutative algebras). The physical counterpart to this transition is the correspondence principle linking quantum to classical mechanics.

The main purpose of these notes is to present an aspect of Poisson geometry which is inherited from the noncommutative side: the notion of Morita equivalence, including the "self-equivalences" known as Picard groups.

In algebra, the importance of Morita equivalence lies in the fact that Morita equivalent algebras have, by definition, equivalent categories of modules. From this it follows that many other invariants, such as cohomology and deformation theory, are shared by all Morita equivalent algebras. In addition, one can sometimes understand the representation theory of a given algebra by analyzing that of a simpler representative of its Morita equivalence class. In Poisson geometry, the role of "modules" is played by Poisson maps from symplectic manifolds to a given Poisson manifold. The simplest such maps are the inclusions of symplectic leaves, and indeed the structure of the leaf space is a Morita invariant. (We will see that this leaf space sometimes has a more rigid structure than one might expect.)

The main theorem of algebraic Morita theory is that Morita equivalences are implemented by bimodules. The same thing turns out to be true in Poisson geometry, with the proper geometric definition of "bimodule".

Here is a brief outline of what follows this introduction.

Chapter 2 is an introduction to Poisson geometry and some of its recent generalizations, including Dirac geometry and "twisted" Poisson geometry in the presence of a "background" closed 3-form. Both of these generalizations are used simultaneously to get a geometric understanding of new notions of symmetry of growing importance in mathematical physics, especially with background 3-forms arising throughout string theory (in the guise of the more familiar closed 2-forms on spaces of curves).

In Chapter 3, we review various flavors of the algebraic theory of Morita equivalence in a way which transfers easily to the geometric case. In fact, some of our examples come from geometry: algebras of smooth functions. Others come from the quantum side: operator algebras.

Chapter 4 is the heart of these notes, a presentation of the geometric Morita theory of Poisson manifolds and the closely related Morita theory of symplectic groupoids. We arrive at this theory via the Morita theory of Lie groupoids in general.

In Chapter 5, we attempt to remedy a defect in the theory of Chapter 4. Poisson manifolds with equivalent (even isomorphic) representation categories may not be Morita equivalent. We introduce refined versions of the representation category (some of which are not really categories!) which do determine the Morita equivalence class. Much of the material in this chapter is new and has not yet appeared in print. (Some of it is based on discussions which came after the PQR Euroschool where this course was presented.)

Along the way, we comment on a pervasive problem in the geometric theory. Many constructions involve forming the leaf space of a foliation, but these leaf spaces are not always manifolds. We make some remarks about the use of differentiable stacks as a language for admitting pathological leaf spaces into the world of smooth geometry.

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Poisson geometry and some generalizations

2.1 Poisson manifolds

Let P be a smooth manifold. A **Poisson structure** on P is an $\mathbb{R}$-bilinear Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(P)$ satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \quad \text{for all } f, g, h \in C^\infty(P). \tag{1}$$

A **Poisson algebra** is a commutative associative algebra which is also a Lie algebra so that the associative multiplication and the Lie bracket are related by (1).

For a function $f \in C^\infty(P)$, the derivation $X_f = \{f, \cdot\}$ is called the **hamiltonian vector field** of f. If $X_f = 0$, we call f a **Casimir function** (see Remark 2.4). It follows from (1) that there exists a bivector field $\Pi \in \mathcal{X}^2(P) = \Gamma(\bigwedge^2 TP)$ such that

$$\{f, g\} = \Pi(df, dg);$$

the Jacobi identity for $\{\cdot,\cdot\}$ is equivalent to the condition $[\Pi, \Pi] = 0$, where $[\cdot,\cdot]$ is the Schouten- Nijenhuis bracket, see e.g. [85].

In local coordinates $(x_1, \cdots, x_n)$, the tensor Π is determined by the matrix

$$\Pi_{ij}(x) = \{x_i, x_j\}. \tag{2}$$

If this matrix is invertible at each x, then Π is called nondegenerate or **symplectic**. In this case, the local matrices $(\omega_{ij}) = (-\Pi_{ij})^{-1}$ define a global 2-form $\omega \in \Omega^2(P) = \Gamma(\bigwedge^2 T^*P)$, and the condition $[\Pi, \Pi] = 0$ is equivalent to $d\omega = 0$.

Example 2.1 (*Constant Poisson structures*)
Let $P = \mathbb{R}^n$, and suppose that the $\Pi_{ij}(x)$ are constant. By a linear

change of coordinates, one can find new coordinates

$$(q_1, \cdots, q_k, p_1, \cdots, p_k, e_1, \cdots, e_l), \quad 2k + l = n,$$

so that

$$\Pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

In terms of the bracket, we have

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

which is the original Poisson bracket in mechanics. In this example, all the coordinates e_i are Casimirs.

Example 2.2 (*Poisson structures on* $\mathbb{R}^2$)

Any smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ defines a Poisson structure in $\mathbb{R}^2 = \{(x_1, x_2)\}$ by

$$\{x_1, x_2\} := f(x_1, x_2),$$

and every Poisson structure on $\mathbb{R}^2$ has this form.

Example 2.3 (*Lie-Poisson structures*)

An important class of Poisson structures are the linear ones. If P is a (finite-dimensional) vector space V considered as a manifold, with linear coordinates $(x_1, \cdots, x_n)$, a linear Poisson structure is determined by constants c_{ij}^k satisfying

$$\{x_i, x_j\} = \sum_{k=1}^{n} c_{ij}^k x_k. \tag{3}$$

(We may assume that $c_{ij}^k = -c_{ji}^k$.) Such Poisson structures are usually called **Lie-Poisson structures**, since the Jacobi identity for the Poisson bracket implies that the c_{ij}^k are the structure constants of a Lie algebra $\mathfrak{g}$, which may be identified in a natural way with V^*. (Also, these Poisson structures were originally introduced by Lie [56] himself.) Note that we may also identify V with $\mathfrak{g}^*$. Conversely, any Lie algebra $\mathfrak{g}$ with structure constants c_{ij}^k defines by (3) a linear Poisson structure on $\mathfrak{g}^*$.

Remark 2.4 (*Casimir functions*)

Deformation quantization of the Lie-Poisson structure on $\mathfrak{g}^*$, see e.g. [10, 45], leads to the universal enveloping algebra $U(\mathfrak{g})$. Elements of the center of $U(\mathfrak{g})$ are known as Casimir elements (or Casimir operators, when a representation of $\mathfrak{g}$ is extended to a representation of $U(\mathfrak{g})$). These correspond to the center of the Poisson algebra of functions on $\mathfrak{g}^*$, hence, by extension, the designation "Casimir functions" for the center of any Poisson algebra.

2.2 Dirac structures

We now introduce a simultaneous generalization of Poisson structures and closed 2-forms. (We will often refer to closed 2-forms as **presymplectic**.)

Each 2-form ω on P corresponds to a bundle map

$$\widetilde{\omega} : TP \to T^*P, \quad \widetilde{\omega}(v)(u) = \omega(v, u). \tag{4}$$

Similarly, for a bivector field $\Pi \in \mathcal{X}^2(P)$, we define the bundle map

$$\widetilde{\Pi} : T^*P \to TP, \quad \beta(\widetilde{\Pi}(\alpha)) = \Pi(\alpha, \beta). \tag{5}$$

The matrix representing $\widetilde{\Pi}$ in the bases (dx_i) and $(\partial/\partial x_i)$ corresponding to local coordinates induced by coordinates $(x_1, \ldots, x_n)$ on P is, up to a sign, just (2). So bivector fields (or 2-forms) are nondegenerate if and only if the associated bundle maps are invertible.

By using the maps in (4) and (5), we can describe both closed 2-forms and Poisson bivector fields as subbundles of $TP \oplus T^*P$: we simply consider the graphs

$$L_\omega := \mathrm{graph}(\widetilde{\omega}), \quad \text{and} \quad L_\Pi := \mathrm{graph}(\widetilde{\Pi}).$$

To see which subbundles of $TP \oplus T^*P$ are of this form, we introduce the following canonical structure on $TP \oplus T^*P$:

1) The symmetric bilinear form $\langle \cdot, \cdot \rangle_+ : TP \oplus T^*P \to \mathbb{R}$,

$$\langle (X, \alpha), (Y, \beta) \rangle_+ := \alpha(Y) + \beta(X). \tag{6}$$

2) The bracket $[\![\cdot, \cdot]\!] : \Gamma(TP \oplus T^*P) \times \Gamma(TP \oplus T^*P) \to \Gamma(TP \oplus T^*P)$,

$$[\![(X, \alpha), (Y, \beta)]\!] := ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha). \tag{7}$$

Remark 2.5 (*Courant bracket*)

The bracket (7) is the non-skew-symmetric version, introduced in [57] (see also [80]), of T. Courant's original bracket [27]. The bundle $TP \oplus T^*P$ together with the brackets (6) and (7) is an example of a **Courant algebroid** [57].

Using the brackets (6) and (7), we have the following result [27]:

Proposition 2.6 *A subbundle* $L \subset TP \oplus T^*P$ *is of the form* $L_\Pi = \mathrm{graph}(\widetilde{\Pi})$ *(resp.* $L_\omega = \mathrm{graph}(\widetilde{\omega})$*) for a bivector field* Π *(resp. 2-form* ω*) if and only if*

i) $TP \cap L = \{0\}$ *(resp.* $L \cap T^*P = \{0\}$*) at all points of* P*;*
ii) L *is maximal isotropic with respect to* $\langle \cdot, \cdot \rangle_+$*;*

furthermore, $[\Pi, \Pi] = 0$ *(resp.* $d\omega = 0$*) if and only if*

iii) $\Gamma(L)$ *is closed under the Courant bracket* (7)*.*

Recall that L being isotropic with respect to $\langle \cdot, \cdot \rangle_+$ means that, at each point of P,

$$\langle (X, \alpha), (Y, \beta) \rangle_+ = 0$$

whenever $(X, \alpha), (Y, \beta) \in L$. Maximality is equivalent to the dimension condition $\mathrm{rank}(L) = \dim(P)$.

A **Dirac structure** on P is a subbundle $L \subset TP \oplus T^*P$ which is maximal isotropic with respect to $\langle \cdot, \cdot \rangle_+$ and whose sections are closed under the Courant bracket (7); in other words, a Dirac structure satisfies conditions *ii*) and *iii*) of Prop. 2.6 but is not necessarily the graph associated to a bivector field or 2-form.

If L satisfies only *ii*), it is called an **almost Dirac structure**, and we refer to *iii*) as the **integrability condition** of a Dirac structure. The next example illustrates these notions in another situation.

Example 2.7 (*Regular foliations*)

Let $F \subseteq TP$ be a subbundle, and let $F^\circ \subset T^*P$ be its annihilator. Then $L = F \oplus F^\circ$ is an almost Dirac structure; it is a Dirac structure if and only if F satisfies the Frobenius condition

$$[\Gamma(F), \Gamma(F)] \subset \Gamma(F).$$

So regular foliations are examples of Dirac structures.

Example 2.8 (*Vector Dirac structures*)

If V is a finite-dimensional real vector space, then a **vector Dirac structure** on V is a subspace $L \subset V \oplus V^*$ which is maximal isotropic with respect to the symmetric pairing (6).[1]

[1] Vector Dirac structures are sometimes called "linear Dirac structures," but we will eschew this name to avoid confusion with linear (i.e. Lie-) Poisson structures. (See Example 2.3)

Let L be a vector Dirac structure on V. Let $\mathrm{pr}_1 : V \oplus V^* \to V$ and $\mathrm{pr}_2 : V \oplus V^* \to V^*$ be the canonical projections, and consider the subspace

$$R := \mathrm{pr}_1(L) \subseteq V.$$

Then L induces a skew-symmetric bilinear form θ on R defined by

$$\theta(X, Y) := \alpha(Y), \tag{8}$$

where $X, Y \in R$ and $\alpha \in V^*$ is such that $(X, \alpha) \in L$.

Exercise
Show that θ is well defined, i.e., (8) is independent of the choice of α.

Conversely, any pair (R, θ), where $R \subseteq V$ is a subspace and θ is a skew-symmetric bilinear form on R, defines a vector Dirac structure by

$$L := \{(X, \alpha), \quad X \in R, \; \alpha \in V^* \text{ with } \alpha|_R = i_X\theta\}. \tag{9}$$

Exercise
Check that L defined in (9) is a vector Dirac structure on V with associated subspace R and bilinear form θ.

Example 2.8 indicates a simple way in which vector Dirac structures can be restricted to subspaces.

Example 2.9 (*Restriction of Dirac structures to subspaces*)
Let L be a vector Dirac structure on V, let $W \subseteq V$ be a subspace, and consider the pair (R, θ) associated with L. Then W inherits the vector Dirac structure L_W from L defined by the pair

$$R_W := R \cap W, \quad \text{and} \quad \theta_W := \iota^*\theta,$$

where $\iota : W \hookrightarrow V$ is the inclusion map.

Exercise
Show that there is a canonical isomorphism

$$L_W \cong \frac{L \cap (W \oplus V^*)}{L \cap W^\circ}. \tag{10}$$

Let (P, L) be a Dirac manifold, and let $\iota : N \hookrightarrow P$ be a submanifold. The construction in Example 2.9, when applied to $T_xN \subseteq T_xP$ for all $x \in P$, defines a maximal isotropic "subbundle" $L_N \subset TN \oplus T^*N$. The problem is that L_N may not be a continuous family of subspaces. When L_N *is* a continuous family, it is a smooth bundle which then automatically satisfies the integrability condition [27, Cor. 3.1.4], so L_N defines a Dirac structure on N.

The next example is a special case of this construction and is one of the original motivations for the study of Dirac structures; it illustrates the connection between Dirac structures and "constraint submanifolds" in classical mechanics.

Example 2.10 (*Momentum level sets*)

Let $J : P \to \mathfrak{g}^*$ be the momentum [2] map for a hamiltonian action of a Lie group G on a Poisson manifold P [59]. Let $\mu \in \mathfrak{g}^*$ be a regular value for J, let G_μ be the isotropy group at μ with respect to the coadjoint action, and consider

$$Q = J^{-1}(\mu) \hookrightarrow P.$$

At each point $x \in Q$, we have a vector Dirac structure on T_xQ given by

$$(L_Q)_x := \frac{L_x \cap (T_xQ \oplus T_x^*P)}{L_x \cap T_xQ^\circ}. \tag{11}$$

To show that L_Q defines a smooth bundle, it suffices to verify that $L_x \cap T_xQ^\circ$ has constant dimension. (Indeed, if this is the case, then $L_x \cap (T_xQ \oplus T_x^*P)$ has constant dimension as well, since the quotient $L_x \cap (T_xQ \oplus T_x^*P)/L_x \cap T_xQ^\circ$ has constant dimension, and this insures that all bundles are smooth.) A direct computation shows that $L_x \cap T_xQ^\circ$ has constant dimension if and only if the stabilizer groups of the G_μ-action on Q have constant dimension, which happens whenever the G_μ-orbits on Q have constant dimension (for instance, when the action of G_μ on Q is locally free). In this case, L_Q is a Dirac structure on Q.

We will revisit this example in Section 2.7.

Remark 2.11 (*Complex Dirac structures and generalized complex geometry*)

Using the natural extensions of the symmetric form (6) and the Courant bracket (7) to $(TP \oplus T^*P) \otimes \mathbb{C}$, one can define a **complex Dirac structure** on a manifold P to be a maximal isotropic *complex* subbundle $L \subset (TP \oplus T^*P) \otimes \mathbb{C}$ whose sections are closed under the

[2] The term "moment" is frequently used instead of "momentum" in this context. In this paper, we will follow the convention, introduced in [61], that "moment" is used only in connection with groupoid actions. As we will see (e.g. in Example 4.16), many momentum maps, even for "exotic" theories, are moment maps as well.

Courant bracket. If a complex Dirac structure L satisfies the condition

$$L \cap \overline{L} = \{0\} \tag{12}$$

at all points of P (here $\overline{L}$ is the complex conjugate of L), then it is called a **generalized complex structure**; such structures were introduced in [43, 46] as a common generalization of complex and symplectic structures.

To see how complex structures fit into this picture, note that an almost complex structure $J : TP \to TP$ defines a maximal isotropic subbundle $L_J \subset (TP \oplus T^*P) \otimes \mathbb{C}$ as the i-eigenbundle of the map

$$(TP \oplus T^*P) \otimes \mathbb{C} \to (TP \oplus T^*P) \otimes \mathbb{C}, \quad (X, \alpha) \mapsto (-J(X), J^*(\alpha)).$$

The bundle L_J completely characterizes J, and satisfies (12); moreover L_J satisfies the integrability condition of a Dirac structure if and only if J is a complex structure.

Similarly, a symplectic structure ω on P can be seen as a generalized complex structure through the bundle $L_{\omega,\mathbb{C}}$, defined as the i-eigenbundle of the map

$$(TP \oplus T^*P) \otimes \mathbb{C} \to (TP \oplus T^*P) \otimes \mathbb{C}, \quad (X, \alpha) \mapsto (\widetilde{\omega}(X), -\widetilde{\omega}^{-1}(\alpha)).$$

Note that, by (12), a generalized complex structure is never the complexification of a real Dirac structure. In particular, for a symplectic structure ω, $L_{\omega,\mathbb{C}}$ is *not* the complexification of the real Dirac structure L_ω of Proposition 2.6.

2.3 Twisted structures

A "background" *closed* 3-form $\phi \in \Omega^3(P)$ can be used to "twist" the geometry of P [48, 69], leading to a modified notion of Dirac structure [80], and in particular of Poisson structure. The key point is to use ϕ to alter the ordinary Courant bracket (7) as follows:

$$[\![(X, \alpha), (Y, \beta)]\!]_\phi := ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha + \phi(X, Y, \cdot)). \tag{13}$$

We now simply repeat the definitions in Section 2.2 replacing (7) by the **ϕ-twisted Courant bracket** (13).

A **ϕ-twisted Dirac structure** on P is a subbundle $L \subset TP \oplus T^*P$ which is maximal isotropic with respect to $\langle \cdot, \cdot \rangle_+$ (6) and for which

$$[\![\Gamma(L), \Gamma(L)]\!]_\phi \subseteq \Gamma(L). \tag{14}$$