1
Introduction

1.1 Preliminaries

A partial differential equation (PDE) describes a relation between an unknown function and its partial derivatives. PDEs appear frequently in all areas of physics and engineering. Moreover, in recent years we have seen a dramatic increase in the use of PDEs in areas such as biology, chemistry, computer sciences (particularly in relation to image processing and graphics) and in economics (finance). In fact, in each area where there is an interaction between a number of independent variables, we attempt to define functions in these variables and to model a variety of processes by constructing equations for these functions. When the value of the unknown function(s) at a certain point depends only on what happens in the vicinity of this point, we shall, in general, obtain a PDE. The general form of a PDE for a function $u(x_1, x_2, \ldots, x_n)$ is

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_{11}}, \ldots) = 0, \tag{1.1}$$

where $x_1, x_2, \ldots, x_n$ are the independent variables, u is the unknown function, and u_{x_i} denotes the partial derivative $\partial u/\partial x_i$. The equation is, in general, supplemented by additional conditions such as initial conditions (as we have often seen in the theory of ordinary differential equations (ODEs)) or boundary conditions.

The analysis of PDEs has many facets. The classical approach that dominated the nineteenth century was to develop methods for finding explicit solutions. Because of the immense importance of PDEs in the different branches of physics, every mathematical development that enabled a solution of a new class of PDEs was accompanied by significant progress in physics. Thus, the method of characteristics invented by Hamilton led to major advances in optics and in analytical mechanics. The Fourier method enabled the solution of heat transfer and wave

propagation, and Green's method was instrumental in the development of the theory of electromagnetism. The most dramatic progress in PDEs has been achieved in the last 50 years with the introduction of numerical methods that allow the use of computers to solve PDEs of virtually every kind, in general geometries and under arbitrary external conditions (at least in theory; in practice there are still a large number of hurdles to be overcome).

The technical advances were followed by theoretical progress aimed at understanding the solution's structure. The goal is to discover some of the solution's properties before actually computing it, and sometimes even without a complete solution. The theoretical analysis of PDEs is not merely of academic interest, but rather has many applications. It should be stressed that there exist very complex equations that cannot be solved even with the aid of supercomputers. All we can do in these cases is to attempt to obtain qualitative information on the solution. In addition, a deep important question relates to the formulation of the equation and its associated side conditions. In general, the equation originates from a model of a physical or engineering problem. It is not automatically obvious that the model is indeed consistent in the sense that it leads to a solvable PDE. Furthermore, it is desired in most cases that the solution will be unique, and that it will be stable under small perturbations of the data. A theoretical understanding of the equation enables us to check whether these conditions are satisfied. As we shall see in what follows, there are many ways to solve PDEs, each way applicable to a certain class of equations. Therefore it is important to have a thorough analysis of the equation before (or during) solving it.

The fundamental theoretical question is whether the problem consisting of the equation and its associated side conditions is well posed. The French mathematician Jacques Hadamard (1865–1963) coined the notion of *well-posedness*. According to his definition, a problem is called well-posed if it satisfies all of the following criteria

1. **Existence** The problem has a solution.
2. **Uniqueness** There is no more than one solution.
3. **Stability** A small change in the equation or in the side conditions gives rise to a small change in the solution.

If one or more of the conditions above does not hold, we say that the problem is *ill-posed*. One can fairly say that the fundamental problems of mathematical physics are all well-posed. However, in certain engineering applications we might tackle problems that are ill-posed. In practice, such problems are unsolvable. Therefore, when we face an ill-posed problem, the first step should be to modify it appropriately in order to render it well-posed.

1.2 Classification

We pointed out in the previous section that PDEs are often classified into different types. In fact, there exist several such classifications. Some of them will be described here. Other important classifications will be described in Chapter 3 and in Chapter 9.

- **The order of an equation**
  The first classification is according to the *order* of the equation. The order is defined to be the order of the highest derivative in the equation. If the highest derivative is of order k, then the equation is said to be of order k. Thus, for example, the equation $u_{tt} - u_{xx} = f(x,t)$ is called a second-order equation, while $u_t + u_{xxxx} = 0$ is called a fourth-order equation.
- **Linear equations**
  Another classification is into two groups: linear versus nonlinear equations. An equation is called *linear* if in (1.1), F is a linear function of the unknown function u and its derivatives. Thus, for example, the equation $x^7 u_x + e^{xy} u_y + \sin(x^2 + y^2)u = x^3$ is a linear equation, while $u_x^2 + u_y^2 = 1$ is a nonlinear equation. The nonlinear equations are often further classified into subclasses according to the type of the nonlinearity. Generally speaking, the nonlinearity is more pronounced when it appears in a higher derivative. For example, the following two equations are both nonlinear:

$$u_{xx} + u_{yy} = u^3, \tag{1.2}$$

$$u_{xx} + u_{yy} = |\nabla u|^2 u. \tag{1.3}$$

  Here $|\nabla u|$ denotes the norm of the gradient of u. While (1.3) is nonlinear, it is still linear as a function of the highest-order derivative. Such a nonlinearity is called *quasilinear*. On the other hand in (1.2) the nonlinearity is only in the unknown function. Such equations are often called *semilinear*.
- **Scalar equations versus systems of equations**
  A single PDE with just one unknown function is called a *scalar equation*. In contrast, a set of m equations with l unknown functions is called a *system* of m equations.

1.3 Differential operators and the superposition principle

A function has to be k times differentiable in order to be a solution of an equation of order k. For this purpose we define the set $C^k(D)$ to be the set of all functions that are k times continuously differentiable in D. In particular, we denote the set of continuous functions in D by $C^0(D)$, or $C(D)$. A function in the set C^k that satisfies a PDE of order k, will be called a *classical* (or *strong*) solution of the PDE. It should be stressed that we sometimes also have to deal with solutions that are not classical. Such solutions are called *weak* solutions. The possibility of weak solutions and their physical meaning will be discussed on several occasions later,

see for example Sections 2.7 and 10.2. Note also that, in general, we are required to solve a problem that consists of a PDE and associated conditions. In order for a strong solution of the PDE to also be a strong solution of the full problem, it is required to satisfy the additional conditions in a smooth way.

Mappings between different function sets are called *operators*. The operation of an operator L on a function u will be denoted by $L[u]$. In particular, we shall deal in this book with operators defined by partial derivatives of functions. Such operators, which are in fact mappings between different C^k classes, are called *differential operators*.

An operator that satisfies a relation of the form

$$L[a_1u_1 + a_2u_2] = a_1L[u_1] + a_2L[u_2],$$

where a_1 and a_2 are arbitrary constants, and u_1 and u_2 are arbitrary functions is called a *linear operator*. A linear differential equation naturally defines a linear operator: the equation can be expressed as $L[u] = f$, where L is a linear operator and f is a given function.

A linear differential equation of the form $L[u] = 0$, where L is a linear operator, is called a *homogeneous equation*. For example, define the operator $L = \partial^2/\partial x^2 - \partial^2/\partial y^2$. The equation

$$L[u] = u_{xx} - u_{yy} = 0$$

is a homogeneous equation, while the equation

$$L[u] = u_{xx} - u_{yy} = x^2$$

is an example of a *nonhomogeneous equation*.

Linear operators play a central role in mathematics in general, and in PDE theory in particular. This results from the important property (which follows at once from the definition) that if for $1 \le i \le n$, the function u_i satisfies the linear differential equation $L[u_i] = f_i$, then the linear combination $v := \sum_{i=1}^n \alpha_i u_i$ satisfies the equation $L[v] = \sum_{i=1}^n \alpha_i f_i$. In particular, if each of the functions $u_1, u_2, \ldots, u_n$ satisfies the homogeneous equation $L[u] = 0$, then every linear combination of them satisfies that equation too. This property is called the *superposition principle*. It allows the construction of complex solutions through combinations of simple solutions. In addition, we shall use the superposition principle to prove uniqueness of solutions to linear PDEs.

1.4 Differential equations as mathematical models

PDEs are woven throughout science and technology. We shall briefly review a number of canonical equations in different areas of application. The fundamental

laws of physics provide a mathematical description of nature's phenomena on a variety of scales of time and space. Thus, for example, very large scale phenomena (astronomical scales) are controlled by the laws of gravity. The theory of electromagnetism controls the scales involved in many daily activities, while quantum mechanics is used to describe phenomena on the atomic scale. It turns out, however, that many important problems involve interaction between a large number of objects, and thus it is difficult to use the basic laws of physics to describe them. For example, we do not fall to the floor when we sit on a chair. Why? The fundamental reason lies in the electric forces between the atoms constituting the chair. These forces endow the chair with high rigidity. It is clear, though, that it is not feasible to solve the equations of electromagnetism (Maxwell's equations) to describe the interaction between such a vast number of objects. As another example, consider the flow of a gas. Each molecule obeys Newton's laws, but we cannot in practice solve for the evolution of an Avogadro number of individual molecules. Therefore, it is necessary in many applications to develop simpler models.

The basic approach towards the derivation of these models is to define new quantities (temperature, pressure, tension,. . .) that describe average macroscopic values of the fundamental microscopic quantities, to assume several fundamental principles, such as conservation of mass, conservation of momentum, conservation of energy, etc., and to apply the new principles to the macroscopic quantities. We shall often need some additional ad-hoc assumptions to connect different macroscopic entities. In the optimal case we would like to start from the fundamental laws and then average them to achieve simpler models. However, it is often very hard to do so, and, instead, we shall sometimes use experimental observations to supplement the basic principles. We shall use x, y, z to denote spatial variables, and t to denote the time variable.

1.4.1 The heat equation

A common way to encourage scientific progress is to confer prizes and awards. Thus, the French Academy used to set up competitions for its prestigious prizes by presenting specific problems in mathematics and physics. In 1811 the Academy chose the problem of heat transfer for its annual prize. The prize was awarded to the French mathematician Jean Baptiste Joseph Fourier (1768–1830) for two important contributions. (It is interesting to mention that he was not an active scientist at that time, but rather the governor of a region in the French Alps – actually a politician!). He developed, as we shall soon see, an appropriate differential equation, and, in addition developed, as we shall see in Chapter 5, a novel method for solving this equation.

The basic idea that guided Fourier was conservation of energy. For simplicity we assume that the material density and the heat capacity are constant in space and time, and we scale them to be 1. We can therefore identify heat energy with temperature. Let D be a fixed spatial domain, and denote its boundary by ∂D. Under these conditions we shall write down the change in the energy stored in D between time t and time $t + \Delta t$:

$$\int_D [u(x, y, z, t + \Delta t) - u(x, y, z, t)]\, \mathrm{d}V = \int_t^{t+\Delta t} \int_D q(x, y, z, t, u) \mathrm{d}V\, \mathrm{d}t - \int_t^{t+\Delta t} \int_{\partial D} \vec{B}(x, y, z, t) \cdot \hat{n} \mathrm{d}S \mathrm{d}t, \quad (1.4)$$

where u is the temperature, q is the rate of heat production in D, $\vec{B}$ is the heat flux through the boundary, $\mathrm{d}V$ and $\mathrm{d}S$ are space and surface integration elements, respectively, and $\hat{n}$ is a unit vector pointing in the direction of the outward normal to ∂D. Notice that the heat production can be negative (a refrigerator, an air conditioner), as can the heat flux.

In general the heat production is determined by external sources that are independent of the temperature. In some cases (such as an air conditioner controlled by a thermostat) it depends on the temperature itself but not on its derivatives. Hence we assume $q = q(x, y, z, t, u)$. To determine the functional form of the heat flux, Fourier used the experimental observation that 'heat flows from hotter places to colder places'. Recall from calculus that the direction of maximal growth of a function is given by its gradient. Therefore, Fourier postulated

$$\vec{B} = -k(x, y, z)\vec{\nabla} u. \quad (1.5)$$

The formula (1.5) is called *Fourier's law of heat conduction*. The (positive!) function k is called the *heat conduction (or Fourier) coefficient*. The value(s) of k depend on the medium in which the heat diffuses. In a homogeneous domain k is expected to be constant. The assumptions on the functional dependence of q and $\vec{B}$ on u are called *constitutive laws*.

We substitute our formula for q and $\vec{B}$ into (1.4), approximate the t integrals using the mean value theorem, divide both sides of the equation by Δt, and take the limit $\Delta t \to 0$. We obtain

$$\int_D u_t \mathrm{d}V = \int_D q(x, y, z, t, u) \mathrm{d}V + \int_{\partial D} k(x, y, z)\vec{\nabla} u \cdot \hat{n} \mathrm{d}S. \quad (1.6)$$

Observe that the integration in the second term on the right hand side is over a different set than in the other terms. Thus we shall use Gauss' theorem to convert

the surface integral into a volume integral:

$$\int_D [u_t - q - \vec{\nabla} \cdot (k\vec{\nabla}u)]\mathrm{d}V = 0, \tag{1.7}$$

where $\vec{\nabla}\cdot$ denotes the divergence operator. The following simple result will be used several times in the book.

Lemma 1.1 *Let $h(x, y, z)$ be a continuous function satisfying $\int_\Omega h(x, y, z)\mathrm{d}V = 0$ for every domain Ω. Then $h \equiv 0$.*

Proof Let us assume to the contrary that there exists a point $P = (x_0, y_0, z_0)$ where $h(P) \neq 0$. Assume without loss of generality that $h(P) > 0$. Since h is continuous, there exists a domain (maybe very small) D_0, containing P and $\epsilon > 0$, such that $h > \epsilon > 0$ at each point in D_0. Therefore $\int_{D_0} h\mathrm{d}V > \epsilon \mathrm{Vol}(D_0) > 0$ which contradicts the lemma's assumption. □

Returning to the energy integral balance (1.7), we notice that it holds for any domain D. Assuming further that all the functions in the integrand are continuous, we obtain the PDE

$$u_t = q + \vec{\nabla} \cdot (k\vec{\nabla}u). \tag{1.8}$$

In the special (but common) case where the diffusion coefficient is constant, and there are no heat sources in D itself, we obtain the classical heat equation

$$u_t = k\Delta u, \tag{1.9}$$

where we use Δu to denote the important operator $u_{xx} + u_{yy} + u_{zz}$. Observe that we have assumed that the solution of the heat equation, and even some of its derivatives are continuous functions, although we have not solved the equation yet. Therefore, in principle we have to reexamine our assumptions a posteriori. We shall see examples later in the book in which solutions of a PDE (or their derivatives) are *not* continuous. We shall then consider ways to provide a meaning for the seemingly absurd process of substituting a discontinuous function into a differential equation. One of the fundamental ways of doing so is to observe that the integral balance equation (1.6) provides a more fundamental model than the PDE (1.8).

1.4.2 Hydrodynamics and acoustics

Hydrodynamics is the physical theory of fluid motion. Since almost any conceivable volume of fluid (whether it is a cup of coffee or the Pacific Ocean) contains a huge number of molecules, it is not feasible to describe the fluid using the law of electromagnetism or quantum mechanics. Hence, since the eighteenth century

scientists have developed models and equations that are appropriate to macroscopic entities such as temperature, pressure, effective velocity, etc. As explained above, these equations are based on conservation laws.

The simplest description of a fluid consists of three functions describing its state at any point in space-time:

- the density (mass per unit of volume) $\rho(x, y, z, t)$;
- the velocity $\vec{u}(x, y, z, t)$;
- the pressure $p(x, y, z, t)$.

To be precise, we must also include the temperature field in the fluid. But to simplify matters, it will be assumed here that the temperature is a known constant. We start with conservation of mass. Consider a fluid element occupying an arbitrary spatial domain D. We assume that matter neither is created nor disappears in D. Thus the total mass in D does not change:

$$\frac{\partial}{\partial t}\int_D \rho \mathrm{d}V = 0. \tag{1.10}$$

The motion of the fluid boundary is given by the component of the velocity $\vec{u}$ in the direction orthogonal to the boundary ∂D. Thus we can write

$$\int_D \frac{\partial}{\partial t}\rho \mathrm{d}V + \int_{\partial D} \rho\vec{u}\cdot\hat{n}\mathrm{d}S = 0, \tag{1.11}$$

where we denoted the unit external normal to ∂D by $\hat{n}$. Using Gauss' theorem we obtain

$$\int_D [\rho_t + \vec{\nabla}\cdot(\rho\vec{u})]\mathrm{d}V = 0. \tag{1.12}$$

Since D is an arbitrary domain we can use again Lemma 1.1 to obtain the mass *transport equation*

$$\rho_t + \vec{\nabla}\cdot(\rho\vec{u}) = 0. \tag{1.13}$$

Next we require the fluid to satisfy the momentum conservation law. The forces acting on the fluid in D are gravity, acting on each point in the fluid, and the pressure applied at the boundary of D by the rest of the fluid outside D. We denote the density per unit mass of the gravitational force by $\vec{g}$. For simplicity we neglect the friction forces between adjacent fluid molecules. Newton's law of motion implies an equality between the change in the fluid momentum and the total forces acting on the fluid. Thus

$$\frac{\partial}{\partial t}\int_D \rho\vec{u}\mathrm{d}V = -\int_{\partial D} p\hat{n}\mathrm{d}s + \int_D \rho\vec{g}\mathrm{d}V. \tag{1.14}$$

Let us interchange again the t differentiation with the spatial integration, and use (1.13) to obtain the integral balance

$$\int_D [\rho \vec{u}_t + \rho(\vec{u} \cdot \vec{\nabla})\vec{u}]\mathrm{d}V = \int_D (-\vec{\nabla} p + \rho \vec{g})\mathrm{d}V. \tag{1.15}$$

From this balance we deduce the PDE

$$\vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} p + \vec{g}. \tag{1.16}$$

So far we have developed two PDEs for three unknown functions $(\rho, \vec{u}, p)$. We therefore need a third equation to complete the system. Notice that conservation of energy has already been accounted for by assuming that the temperature is fixed. In fact, the additional equation does not follow from a conservation law, rather one imposes a constitutive relation (like Fourier's law from the previous subsection). Specifically, we postulate a relation of the form

$$p = f(\rho), \tag{1.17}$$

where the function f is determined by the specific fluid (or gas). The full system comprising (1.13), (1.16) and (1.17) is called the *Euler fluid flow equations*. These equations were derived in 1755 by the Swiss mathematician Leonhard Euler (1707–1783).

If one takes into account the friction between the fluid molecules, the equations acquire an additional term. This friction is called *viscosity*. The special case of viscous fluids where the density is essentially constant is of particular importance. It characterizes, for example, most phenomena involving the flow of water. This case was analyzed first in 1822 by the French engineer Claude Navier (1785–1836), and then studied further by the British mathematician George Gabriel Stokes (1819–1903). They derived the following set of equations:

$$\rho(\vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u}) = \mu \Delta \vec{u} - \vec{\nabla} p, \tag{1.18}$$

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{1.19}$$

The parameter μ is called the fluid's viscosity. Notice that (1.18)–(1.19) form a quasilinear system of equations. The Navier–Stokes system lies at the foundation of hydrodynamics. Enormous computational efforts are invested in solving them under a variety of conditions and in a plurality of applications, including, for example, the design of airplanes and ships, the design of vehicles, the flow of blood in arteries, the flow of ink in a printer, the locomotion of birds and fish, and so forth. Therefore it is astonishing that the well-posedness of the Navier–Stokes equations has not yet been established. Proving or disproving their well-posedness is one of the most

important open problems in mathematics. A prize of one million dollars awaits the person who solves it.

An important phenomenon described by the Euler equations is the propagation of sound waves. In order to construct a simple model for sound waves, let us look at the Euler equations for a gas at rest. For simplicity we neglect gravity. It is easy to check that the equations have a solution of the form

$$\begin{aligned} \vec{u} &= 0, \\ \rho &= \rho_0, \\ p &= p_0 = f(\rho_0), \end{aligned} \tag{1.20}$$

where ρ_0 and p_0 are constants describing uniform pressure and density. Let us perturb the gas by creating a localized pressure (for example by producing a sound out of our throats, or by playing a musical instrument). Assume that the perturbation is small compared with the original pressure p_0. One can therefore write

$$\begin{aligned} \vec{u} &= \epsilon \vec{u}^1, \\ \rho &= \rho^0 + \epsilon \rho^1, \\ p &= p_0 + \epsilon p^1 = f(\rho^0) + \epsilon f'(\rho^0)\rho^1, \end{aligned} \tag{1.21}$$

where we denoted the perturbation to the density, velocity and pressure by $\vec{u}^1$, ρ^1, and p^1, respectively, ϵ denotes a small positive parameter, and we used (1.17). Substituting the expansion (1.21) into the Euler equations, and retaining only the terms that are linear in ϵ, we find

$$\begin{aligned} \rho^1_t + \rho_o \vec{\nabla} \cdot \vec{u}^1 &= 0, \\ \vec{u}^1_t + \frac{1}{\rho^0} \vec{\nabla} p^1 &= 0. \end{aligned} \tag{1.22}$$

Applying the operator $\vec{\nabla}\cdot$ to the second equation in (1.22), and substituting the result into the time derivative of the first equation leads to

$$\rho^1_{tt} - f'(\rho^0)\Delta \rho^1 = 0. \tag{1.23}$$

Alternatively we can use the linear relation between p^1 and ρ^1 to write a similar equation for the pressure

$$p^1_{tt} - f'(\rho^0)\Delta p^1 = 0. \tag{1.24}$$

The equation we have obtained is called a *wave equation*. We shall see later that this equation indeed describes waves propagating with speed $c = \sqrt{f'(\rho^0)}$. In particular, in the case of waves in a long narrow tube, or in a long and narrow tunnel, the pressure