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Analytic Methods for Diophantine Equations
and Diophantine Inequalities

Harold Davenport was one of the truly great mathematicians of the twentieth century. Based on lectures he gave at the University of Michigan in the early 1960s, this book is concerned with the use of analytic methods in the study of integer solutions to Diophantine equations and Diophantine inequalities. It provides an excellent introduction to a timeless area of number theory that is still as widely researched today as it was when the book originally appeared. The three main themes of the book are Waring's problem and the representation of integers by diagonal forms, the solubility in integers of systems of forms in many variables, and the solubility in integers of diagonal inequalities.

For the second edition of the book a comprehensive foreword has been added in which three leading experts describe the modern context and recent developments. A complete list of references has also been added.

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Second edition

H. Davenport

This edition edited and prepared for publication by

T. D. Browning
Oxford University

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Foreword

Waring's problem: Chapters 1–10

When Davenport produced these lecture notes there had been very little progress on Waring's problem since important work by Davenport and Vinogradov something like a quarter of a century earlier, and the main interest was to report on the more recent work on forms as described in the later chapters. Indeed there was a generally held view, with regard to Waring's problem at least, that they had extracted everything that could be obtained reasonably by the Hardy–Littlewood method and that the method was largely played out. Moreover, the material on Waring's problem was not intended, in general, to be state of the art, but rather simply an introduction to the Hardy–Littlewood method, with a minimum of fuss by a masterly expositor, which could then be developed as necessary for use in the study of the representation of zero by general integral forms, especially cubic forms, in the later chapters. There is no account of Davenport's own fundamental work on Waring's problem, namely $G(4) = 16$ (Davenport [18]), $G(5) \leq 23$, $G(6) \leq 36$ (Davenport [19]), nor of Vinogradov's [94] $G(k) \leq 2k \log k + o(k \log k)$ for large k or Davenport's proof [17] that almost all natural numbers are the sum of four positive cubes. Nor, on a more technical level, was any attempt made to obtain more refined versions of Lemmas 4.2 and 9.2, estimates for the generating function $T(\alpha)$ on the major arcs, such as those due to Davenport and Heilbronn [25] or Hua [50], although such refinements can be very helpful in applications.

In the last twenty years there has been a good deal of progress on Waring's problem. Methods of great flexibility, inspired by some of the ideas stemming from the researches of Hardy and Littlewood, Davenport, and Vinogradov have been developed which have permitted the retention of

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many of the wrinkles introduced in the earlier methods. The beginnings of a glimmer of some of these seminal ideas can be seen in Lemmas 9.4 and 9.5.

The asymptotic formula for the number of representations of a large natural number n as the sum of at most s kth powers established in Theorem 4.1 when $s \geq 2^k + 1$ was state of the art for $3 \leq k \leq 10$, but for larger k methods due to Vinogradov were superior (see Theorem 5.4 of Vaughan [86]). The current state of play is that the asymptotic formula is known to hold when $s \geq 2^k$ $(k = 3, 4, 5)$ (Vaughan [82, 84]), $s \geq 7.2^{k-3}$ $(k = 6, 7, 8)$ (Boklan [8], following Heath-Brown [43]), and $s \geq s_1(k)$ where $s_1(k) = k^2\big(\log k + \log\log k + O(1)\big)$ $(k \geq 9)$ (Ford [32]). The discussion in the Note in Chapter 3 in the case $k = 3$ is still relevant today. Although the asymptotic formula for sums of eight cubes is now established the classical convexity bound was not improved in the exponent when $2 < m < 4$. The core of the argument of Vaughan [82] is extremely delicate and leads only to

$$\int_{\mathfrak{m}} |T(\alpha)|^8 d\alpha \ll P^5(\log P)^{-\gamma}$$

for a positive constant γ and a suitable set of minor arcs $\mathfrak{m}$. However Hooley [47] has shown under the (unproven) Riemann Hypothesis for a certain Hasse–Weil L-function that

$$\int_{\mathfrak{m}} |T(\alpha)|^6 d\alpha \ll P^{3+\varepsilon}$$

and this in turn implies the asymptotic formula for sums of seven cubes. Unfortunately it is not even known whether the L-function has an analytic continuation into the critical strip.

For $G(k)$ the best results that we currently have are $G(3) \leq 7$ (Linnik [57, 59]), $G(4) = 16$ Davenport [18], $G(5) \leq 17$, $G(7) \leq 33$, (Vaughan and Wooley [89]), $G(6) \leq 21$ (Vaughan and Wooley [88]), $G(8) \leq 42$ (Vaughan and Wooley [87]), $G(9) \leq 50$, $G(10) \leq 59$, $G(11) \leq 67$, $G(12) \leq 76$, $G(13) \leq 84$, $G(14) \leq 92$, $G(15) \leq 100$, $G(16) \leq 109$, $G(17) \leq 117$, $G(18) \leq 125$, $G(19) \leq 134$, $G(20) \leq 142$ (Vaughan and Wooley [90]), and $G(k) \leq s_2(k)$ where $s_2(k) = k\big(\log k + \log\log k + O(1)\big)$ (Wooley [98]) in general. Let $G^{\#}(4)$ denote the smallest positive s such that whenever $1 \leq r \leq s$ every sufficiently large n in the residue class r modulo 16 is the sum of at most s fourth powers. Then, in fact, Davenport showed that $G^{\#}(4) \leq 14$ and we now can prove (Vaughan [85]) that $G^{\#}(4) \leq 12$. Linnik's work on Waring's problem for cubes does not use the Hardy–Littlewood method, but instead is based on the

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theory of ternary quadratic forms. Watson [95] gave a similar but simpler proof. However these proofs give relatively poor information about the number of representations as a sum of seven cubes. As part of the recent progress we now have proofs via the Hardy–Littlewood method (e.g. Vaughan [85]) which give lower bounds of the expected correct order of magnitude for the number of representations. Davenport gives no indication of what he might have believed the correct value of $G(k)$ to be. The simplest guess is that

$$G(k) = \max\{k+1, \Gamma(k)\}$$

where $\Gamma(k)$ is as defined in the paragraph just prior to Theorem 5.1. This would imply that for $k \geq 3$, $G(k) = 4k$ when $k = 2^l$ and $k+1 \leq G(k) \leq \frac{3}{2}k$ when $k \neq 2^l$.

With regard to Lemma 9.2 and the Note after the proof, we now know that under the less stringent hypothesis $(q,a) = 1$, $q|\beta| \leq \frac{1}{2k}P^{1-k}$, $\alpha = \beta + a/q$ we have the stronger estimate

$$T(\alpha) = q^{-1}S_{a,q}I(\beta) + O\big(q^{\frac{1}{2}+\varepsilon}\big).$$

Moreover with only the hypothesis $(q,a) = 1$ we have

$$T(\alpha) = q^{-1}S_{a,q}I(\beta) + O\big(q^{\frac{1}{2}+\varepsilon}(1+P^k|\beta|)^{\frac{1}{2}}\big).$$

See Theorem 4.1 of Vaughan [86]. The latter result enables a treatment to be given for cubes in which all the arcs are major arcs.

For a modern introduction to the Hardy–Littlewood method and some of the more recent developments as applied to Waring's problem see Vaughan [86], and for a comprehensive survey of Waring's problem see Vaughan and Wooley [91].

Chapter 7 is concerned with the solubility, given a sequence $\{c_j\}$ of natural numbers, of the equation

$$c_1x_1^k + \cdots + c_sx_s^k = N \tag{1}$$

for large natural numbers N, and is really a warm-up for Chapters 8 and 10. For an infinite set of N there may not be solutions, however large one takes s to be, but the obstruction is purely a local one. Any of the various forms of the Hardy–Littlewood method which have been developed for treating Waring's problem are readily adjusted to this slightly more general situation and, with the corresponding condition on s, lead to an approximate formula for the number of solutions counted. This will lead to a positive lower bound for the number of solutions for any large N for which the singular series is bounded away from 0.

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Davenport gives a brief outline of the minor changes in the argument which have to be made in adapting the method, and the remainder of the chapter is devoted to showing that the above condition on the singular series is essentially equivalent to the expected local solubility condition.

In Chapters 8 and 10, Davenport adapts the method to treat

$$c_1 x_1^k + \cdots + c_s x_s^k = 0 \tag{2}$$

where now the c_j can be integers, and not all the same sign when k is even. Of course this has a solution, and so the main point of interest is to establish the existence of integral solutions in which not all the x_j are 0. This can be considered to be the first special case of what was the main concern of these notes, namely to investigate the non-trivial representation of 0 by general forms and systems of forms. In Chapter 8 the simplest version of the Hardy–Littlewood method developed in the previous chapters is suitably adapted. This requires quite a large value of s to ensure a solution. In Chapter 10 this requirement is relaxed somewhat by adapting the variant of Vinogradov's argument used to treat Waring's problem in Chapter 9. Although the argument of Chapter 10 is relatively simple it is flawed from a philosophical point of view in that as well as the local solubility of (2) there needs to be a discussion of the local solubility of (1) with N non-zero, which, of course, really should not be necessary. This could have been avoided, albeit with some complications of detail. The question of the size of s to ensure a non-trivial solution to (2) had some independent interest as Davenport and Lewis [27] had shown that $k^2 + 1$ variables suffice for the singular series to be bounded away from 0, and when $k + 1$ is prime there are equations in k^2 variables with no non-trivial solution. Moreover they had also shown, via the Hardy–Littlewood method, that (2) is soluble when $s \geq k^2 + 1$ and either $k \leq 6$ or $k \geq 18$. Later in Vaughan [81] $(11 \leq k \leq 17)$, [83] $(7 \leq k \leq 9)$ and [85] $(k = 10)$ this gap was removed. The methods of Vaughan and Wooley mentioned in connection with Waring's problem when adapted show that far fewer variables suffice for a non-trivial solution to (2) provided that the corresponding singular series is bounded away from 0, and this is essentially equivalent to a local solubility condition.

In the later chapters the Hardy–Littlewood method is adapted in various, sometimes quite sophisticated, ways. However, the only place where any of the main results of the first 10 chapters is applied directly is the use of Theorem 8.1 (or Theorem 10.1) in the proof of Birch's theorem in Chapter 11. Later Birch [7] gave a completely elementary proof, based

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partly on methods of Linnik [58], of a result similar to Theorem 8.1 which can be used in its place.

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Forms in many variables: Chapters 11–19

Let $F(x_1, \ldots, x_n)$ be a form of degree d with integer coefficients. When $d \geq 3$, the question of whether the equation $F(x_1, \ldots, x_n) = 0$ has a non-trivial integer solution is extremely natural, extremely general, and extremely hard. However for quadratic forms a complete answer is given by the Hasse–Minkowski Theorem, which states that there is a non-trivial solution if and only if there is such a solution in $\mathbb{R}$ and in each p-adic field $\mathbb{Q}_p$. Such a result is known to be false for higher degree forms, as Selmer's example

$$3x_1^3 + 4x_2^3 + 5x_3^3 = 0$$

shows. None the less the hope remains that if the number of variables is not too small we should still have a 'local-to-global' principle, of the type given by the Hasse–Minkowski Theorem.

It transpires that the p-adic condition holds automatically if the number of variables n is sufficiently large in terms of the degree. This was shown by Brauer [9], whose argument constitutes the first general method for such problems. The line of attack uses multiply nested inductions, and in consequence the necessary number of variables is very large. It was conjectured by Artin that $d^2 + 1$ variables always suffice, there being easy examples of forms in d^2 variables with only trivial p-adic solutions. However many counter-examples have subsequently been discovered. The first of these, due to Terjanian [80], involves a quartic form in 18 variables, with no non-trivial 2-adic solution. There are no known counter-examples involving forms of prime degree, and in this case it remains an open question whether or not Artin's conjecture holds.

There are various alternatives to Brauer's induction approach for the p-adic problem. Davenport presents one of these for the case $d = 3$ in Chapter 18, establishing the best possible result, namely that p-adic solutions always exist when $n \geq 10$. For $d \geq 4$ such approaches work well only when p is large enough. Thus Leep and Yeomans [55] have shown that $p \geq 47$ suffices for $d = 5$. In the general case Ax and Kochen [1] showed that $d^2 + 1$ variables always suffice for the p-adic problem,

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when p is sufficiently large compared with d. The Ax–Kochen proof is remarkable for its use of methods from mathematical logic. For small primes other lines of argument seem to be needed, and Wooley [100] has re-visited the Brauer induction approach to establish that $d^{2^d}+1$ variables suffice for every field $\mathbb{Q}_p$. It remains a significant open problem to get bounds of a reasonable size, below 1000 say, for the cases $d=4$ and $d=5$.

The problem for forms over $\mathbb{Q}$, rather than $\mathbb{Q}_p$, is distinctly different. For forms of even degree there is no value of n which will ensure the existence of a non-trivial integer solution, as the example

$$x_1^d+\cdots+x_n^d=0$$

shows. Thus the original Brauer induction argument cannot be applied to $\mathbb{Q}$, since it involves an induction over the degree. However Birch [5] was able to adapt the induction approach so as to use forms of odd degree only, and hence to show that for any odd integer $d\geq 1$ there is a corresponding $n(d)$ such that $F(x_1,\ldots,x_n)=0$ always has a non-trivial solution for $n\geq n(d)$. This work is described by Davenport in Chapter 11. A rather slicker account is now available in the book by Vaughan [86, Chapter 9]. Although the values of $n(d)$ produced by Birch's work were too large to write down, more reasonable estimates have been provided by Wooley [99], by a careful adaptation of Birch's approach.

Davenport's own major contribution to the area was his attack on cubic forms, via the circle method. The natural application of Weyl's method, as described in Chapter 13, leads to a system of Diophantine inequalities involving bilinear forms. The key result in this context is Lemma 13.2. By using techniques from the geometry of numbers, Davenport was able to convert these inequalities into equations. In his first two papers on the subject [20, 21] these equations were used to deduce that F must represent a form of the type $a_1x_1^3+F'(x_2,\ldots,x_m)$ for some $m<n$. This process is somewhat wasteful, since $n-m$ variables are effectively discarded. By repeated applications of the above principle Davenport was able to reduce consideration to diagonal forms. Davenport's third paper [22] treats the bilinear equations in a more geometrical way, which is presented in Chapter 14. This approach is much more efficient, since no variables are wasted. A straightforward application of this third method shows that $F=0$ has a non-trivial solution for any cubic form in 17 or more variables, and this is the result given as Theorem 18.1. However in [22] a slight refinement is used to show that 16 variables suffice. It is perhaps worth emphasizing the slightly unusual logical

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structure of the proof. The main goal is to prove an asymptotic formula for the number of solutions in a box of side P. Davenport achieves this, providing that the number of solutions to the aforementioned bilinear equations does not grow too rapidly. The arguments used to handle this latter issue lead to two alternatives: either the number of solutions to the bilinear equations is indeed suitably bounded, or the original cubic form has a non-trivial integer zero for geometric reasons. In either case the cubic form has a non-trivial integer zero. One consequence of all this is that one does not obtain an asymptotic formula in every case. The form

$$x_1^3 + x_2(x_3^2 + \cdots + x_n^2)$$

vanishes whenever $x_1 = x_2 = 0$, so that there are $\gg P^{n-2}$ solutions in a box of side P. This example shows that one cannot in general expect an asymptotic formula of the type mentioned in connection with Theorem 17.1.

The 16 variable result is arguably one of Davenport's finest achievements, and it remains an important challenge to show that 15 variables, say, are in fact enough. Davenport's approach has been vastly generalized by Schmidt [77] so as to apply to general systems of forms of arbitrary degree. For a single form $F(x_1, \ldots, x_n)$ the result may be expressed in terms of the invariant $h(F)$ defined as the smallest integer h for which one can write

$$F(\mathbf{x}) = G_1(\mathbf{x})H_1(\mathbf{x}) + \cdots + G_h(\mathbf{x})H_h(\mathbf{x})$$

with non-constant forms G_i, H_i having rational coefficients. An inspection of Davenport's argument for cubic forms in 16 variables then establishes the standard Hardy–Littlewood asymptotic formula for any cubic form with $h(F) \geq 16$. When $h(F) \leq 15$ and $n \geq 16$ the form F still has a non-trivial integer zero, since one can take the forms $H_i(\mathbf{x})$ to be linear and use a common zero of $H_1, \ldots, H_h$. In his generalization Schmidt was able to obtain an explicit function $n(d)$ such that the Hardy–Littlewood formula holds for any form of degree d having $h(F) \geq n(d)$. In order to deal with forms for which $h(F) < n(d)$ one is led to an induction argument involving systems of forms. Thus if one starts with a single form of degree $d = 5$ one wants to know about zeros of systems of cubic forms. In this connection Schmidt proved in a separate investigation [76] that a system of r cubic forms with integer coefficients has a non-trivial integer zero if there are at least $(10r)^5$ variables.

Davenport's result was generalized in another direction by Pleasants

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[67], who showed that the result remains true if the coefficients of the form F, and the solutions $(x_1, \ldots, x_n)$, are allowed to lie in an algebraic number field. In this wider setting 16 variables still suffice.

If one assumes the form F to be non-singular, which is the generic case, one can show (Heath-Brown [42]) that 10 variables suffice. Here the number 10 is best possible, since there exist forms in 9 variables with no non-trivial p-adic zeros. However Hooley [45] has sharpened the above result to establish the local-to-global principle for non-singular cubic forms in $n \geq 9$ variables. These works use the Hardy–Littlewood method, but instead of employing Weyl's inequality they depend on the Poisson summation formula and estimates for 'complete' exponential sums. Complete exponential sums involving a non-singular form can be estimated very efficiently via Deligne's Riemann Hypothesis for varieties over finite fields, but the methods become less effective as the dimension of the singular locus grows. Deligne's bounds handle sums to prime, or square-free, moduli, but sums to prime power moduli remain a considerable problem. The treatment of these in [42] uses exactly the same bilinear forms as were encountered by Davenport [22], but since F is now non-singular the techniques of Birch [6] can be used to advantage. Heath-Brown [42] establishes an asymptotic formula for the number of solutions in a suitable region. However the argument in Hooley [45] has a structure somewhat analogous to Davenport's, in that one only gets an asymptotic formula under a certain geometric condition. When the condition fails there are integer points for other reasons. (This defect was later circumvented by Hooley [46].) In its simplest guise the above methods would handle non-singular cubic forms in 13 or more variables. However this may be reduced to 10 through the use of Kloosterman's refinement of the circle method. In order to handle forms in nine variables Hooley adopts a distinctly more subtle analysis, designed to save just a power of $\log P$, when considering points in a box of side P.

The work of Birch [6], summarized in Chapter 19, is most easily described by seeing how it applies to a single form F. When F is non-singular Birch is able to establish an asymptotic formula as soon as $n > (d-1)2^d$, providing that the singular series and integral are positive. For $d = 3$ this is weaker than the result of Hooley [45], but the method works for arbitrary values of d. In fact subsequent investigations have failed to improve on Birch's result for any value of $d > 3$. Birch's argument is based on Weyl's inequality, and leads to a system of multilinear equations analogous to the bilinear ones in Davenport's work. These are handled by a different technique from that used by

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Davenport, which is simpler and more obviously geometric, but which requires information about the singularities of F.

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Diophantine inequalities: Chapter 20

In the final chapter, Davenport provides an exposition of his groundbreaking 1946 joint work with Heilbronn [26]. They demonstrated how to adapt the Hardy–Littlewood method to yield results on Diophantine inequalities. Since their publication, numerous results have been proved with their technique, now commonly referred to as the Davenport–Heilbronn method.

Suppose that s is an integer with $s \geq 5$ and that $\lambda_1, \ldots, \lambda_s$ are real numbers, not all of the same sign, and not all in rational ratio. The chapter consists of a proof that given any positive real number C, there exists a non-trivial integer solution $\mathbf{x} = (x_1, \ldots, x_s)$ of the Diophantine inequality

$$\left|\lambda_1 x_1^2 + \cdots + \lambda_s x_s^2\right| < C. \tag{3}$$

As Davenport notes, the result has a straightforward extension to the case in which the squares are replaced by kth powers and the number of variables is at least $2^k + 1$. If k is odd, the sign condition is of course unnecessary.

The proof is a clever adaptation of the Hardy–Littlewood method. One estimates, for some large positive P, the number of solutions of (3) where the integers x_i satisfy $|x_i| \leq P$. Rather than integrating over a unit interval as in the Hardy–Littlewood method, one integrates over the real line against a suitable decaying kernel. Instead of multiple major arcs, here the major contribution comes from an interval centred around zero, while the most difficult region to bound consists of a subset of numbers of intermediate size. The contribution to this latter region is treated using the hypothesis that one of the ratios is irrational.

In the lecture notes, Davenport conjectures that (3) is non-trivially soluble even for $s \geq 3$, and in a separate comment notes that a natural question is whether the result can be generalized to the case of indefinite quadratic forms that are not necessarily diagonal and discusses some work by Birch, Davenport and Ridout (see [29]). In fact, Margulis [60] answered both questions positively, establishing the non-trivial solubility

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of

$$|Q(x_1, \ldots, x_s)| < \varepsilon$$

for general indefinite quadratic forms $Q(\mathbf{x})$, for any $\varepsilon > 0$, assuming $s \geq 3$ and that the coefficients of Q are not all in rational ratio. This established the Oppenheim conjecture, as it implies that the values of such a form at integral points are dense on the real line. We note that Margulis' proof uses techniques different from the Hardy–Littlewood method.

Concerning forms of higher degree, Davenport mentions a result that Pitman [66] gave on cubic forms, but remarks that proving similar results for forms of higher odd degree seems to involve a 'difficulty of principle'. Schmidt, in a sequence of papers [73, 74, 75], provided the key result needed to resolve this difficulty. His work builds on a combination of the Davenport–Heilbronn method and a diagonalization procedure that yields a proof that any system of general Diophantine inequalities of odd degree and sufficiently many variables has a solution. More precisely, he showed that given odd positive integers $d_1, \ldots, d_R$, there exists a constant $C(d_1, \ldots, d_R)$ depending only on $d_1, \ldots, d_R$ such that given any real forms $F_1, \ldots, F_R$ in s variables, of respective degrees $d_1, \ldots, d_R$, where $s \geq C(d_1, \ldots, d_R)$, and given $\varepsilon > 0$, there exists a non-trivial integer vector $\mathbf{x}$ such that

$$|F_1(\mathbf{x})| < \varepsilon, \quad |F_2(\mathbf{x})| < \varepsilon, \quad \ldots \quad , \quad |F_R(\mathbf{x})| < \varepsilon.$$

There are numerous results which give lower bounds such as $C(d_1, \ldots, d_R)$ for particular types of forms, of which we mention only two. Brüdern and Cook [11] produced such a result for systems of diagonal forms, under certain conditions on the coefficients, and Nadesalingam and Pitman [62] have given an explicit lower bound for systems of R diagonal cubic forms.

One can also ask about inequalities involving general positive definite forms with coefficients not all in rational ratio. We certainly do not expect the values at integral points to be dense on the real line; thus the relevant question, asked by Estermann, is whether the gaps between these values tend to zero as the values tend to infinity, provided that the number of variables is sufficiently large. For diagonal quadratic forms, Davenport and Lewis [28] noted that this follows readily from a result of Jarník and Walfisz [51], if the number of variables s is at least 5. In their paper, Davenport and Lewis gave a step toward answering the gaps question for general positive definite quadratic forms $Q(\mathbf{x})$ in s

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variables. Their methods essentially show, as Cook and Raghavan [15] demonstrate, that for such forms, given s sufficiently large and given $\varepsilon > 0$, then for any sufficiently large integral point $\mathbf{x}_0$, there are many integral points $\mathbf{x}$ for which one has $|Q(\mathbf{x}) - Q(\mathbf{x}_0)| < \varepsilon$, where the notion of many can be defined precisely. In 1999, Bentkus and Götze [3] resolved the gaps question with powerful new techniques, which Götze [36] consequently improved upon. These results together establish that for $s \geq 5$ and for any positive definite quadratic form Q in s variables, with coefficients not all in rational ratio, the differences between successive values of Q at integral points tend to zero as the values approach infinity. Their methods have given rise to much new work on Diophantine inequalities. Additionally, we note that some workers have considered special types of inhomogeneous polynomials of higher degree, including Brüdern [10], Bentkus and Götze [4] and Freeman [34].

Since Davenport and Heilbronn's work, there have been many improvements of the lower bound on s required to guarantee non-trivial solubility of diagonal Diophantine inequalities of degree k. For each positive integer k, let $G_{\text{ineq}}(k)$ denote the smallest positive integer s_0 such that for all $s \geq s_0$, and for all indefinite diagonal forms $\lambda_1 x_1^k + \cdots + \lambda_s x_s^k$ with coefficients not all in rational ratio, and for all $\varepsilon > 0$, there is a non-trivial integral solution of

$$\left|\lambda_1 x_1^k + \cdots + \lambda_s x_s^k\right| < \varepsilon. \tag{4}$$

As Davenport remarks, Davenport and Roth [30] provided an improvement; they showed that there exists a constant $C_1 > 0$ such that

$$G_{\text{ineq}}(k) \leq C_1 k \log k.$$

In fact, the Davenport–Heilbronn method is sufficiently flexible so that bounds for inequalities roughly parallel bounds given by work on Waring's problem. In particular, for large k, one has

$$G_{\text{ineq}}(k) \leq k(\log k + \log\log k + 2 + o(1)). \tag{5}$$

(See [101] for a statement of this result.) We note that in many cases, for example the work of Baker, Brüdern and Wooley [2] for $k = 3$, achieving the same bound as that for $G(k)$ required extra effort. Recent work of Wooley [101] shows that bounds for $G(k)$ generally, with some exceptions, apply as bounds for $G_{\text{ineq}}(k)$.

As Davenport notes, the proof in Chapter 20 only applies to a sequence of large P, where the sequence depends on the rational approximation properties of the ratios of the coefficients. In many applications of the

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Hardy–Littlewood method, one obtains an asymptotic formula for the number of integral solutions for all positive P with not much more effort than is required to establish solubility. For example, for indefinite diagonal forms with coefficients nonzero and not all in rational ratio, and for positive P, and s sufficiently large in terms of k, we would expect that the number $N(P)$ of integral solutions $\mathbf{x}$ of (4) with $|x_i| \leq P$ for $1 \leq i \leq s$ satisfies

$$N(P) = C(s, k, \lambda_1, \ldots, \lambda_s)\varepsilon P^{s-k} + o\left(P^{s-k}\right), \tag{6}$$

where $C(s, k, \lambda_1, \ldots, \lambda_s)$ is a positive constant depending only on s, k and the coefficients λ_i. However, the proof of Davenport and Heilbronn (with some minor technical modifications) allows one to give asymptotic formulae for diagonal Diophantine inequalities for essentially only an infinite sequence of large P. In their paper, Bentkus and Götze [3] establish the appropriate analogue of (6) for general positive definite quadratic forms with coefficients not all in rational ratio, for all positive P; although their proofs are not phrased in the language of the Davenport–Heilbronn method, the ideas are similar. By adapting their work, Freeman [33, 35] was able to prove the existence of an asymptotic formula such as (6) for indefinite diagonal forms of degree k for all positive P. Wooley [101] has considerably simplified and improved this work, using clever ideas to reduce the number of variables needed to guarantee the existence of asymptotic formulae.

In particular, for the existence of asymptotic formulae for large k, one can establish results similar to (5); if we define $G_{\text{asymp}}(k)$ analogously to $G_{\text{ineq}}(k)$, one has

$$G_{\text{asymp}}(k) \leq k^2 \left(\log k + \log\log k + O(1)\right).$$

Finally, we note that Eskin, Margulis and Mozes [31], using techniques different from the Davenport–Heilbronn method, in fact earlier proved the existence of asymptotic formulae of the expected kind for the case of general indefinite quadratic forms in at least four variables with coefficients not all in rational ratio, and signature not equal to $(2, 2)$.

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Editorial preface

Like many mathematicians I first came into contact with number theory through Davenport's book *The Higher Arithmetic* [23]. It was difficult not to be struck by his command of the subject and wonderful expository style. This basic textbook is now into its seventh edition, whilst at a more advanced level, a third edition of Davenport's *Multiplicative Number Theory* [24] has recently appeared. It is fair to say therefore that Davenport still holds considerable appeal to mathematicians worldwide. On discovering that Davenport had also produced a rather less well-known set of lecture notes treating an area of substantial current interest, I was immediately compelled to try and get it back into print. In doing so, I have tried to preserve in its original format as much of the material as possible, and have merely removed errors that I encountered along the way.

As the title indicates, this book is concerned with the use of analytic methods in the study of integer solutions to certain polynomial equations and inequalities. It is based on lectures that Davenport gave at the University of Michigan in the early 1960s. This analytic method is usually referred to as the 'Hardy–Littlewood circle method', and its power is readily demonstrated by the diverse range of number theoretic problems that can be tackled by it. The first half of the book is taken up with a discussion of the method in its most classical setting: Waring's problem and the representation of integers by diagonal forms. In Chapters 11–19, Davenport builds upon these foundations by showing how the method can sometimes be adapted to handle integer solutions of general systems of homogeneous polynomial equations. Finally, in Chapter 20 Davenport presents an account of work carried out by himself and Heilbronn in the setting of Diophantine inequalities. Even more so than with his

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other books, these lecture notes reflect Davenport's extensive influence in the subject area and his deep knowledge pertaining to it.

This edition of Davenport's lecture notes has been considerably enriched by the provision of a foreword, the main purpose of which is to place a modern perspective on the state of knowledge described in the lecture notes. I am extremely grateful to Professor Freeman, Professor Heath-Brown and Professor Vaughan for lending their authority to this project. I also wish to thank Lillian Pierce and Luke Woodward for all of their hard work in helping me transcribe Davenport's original lecture notes into LaTeX. Finally it is a pleasure to express my gratitude both to James Davenport at Bath University and to David Tranah at Cambridge University Press for sharing my enthusiasm in bringing these lecture notes to the attention of a wider mathematical audience.

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