FOUNDATIONS OF DYNAMIC ECONOMIC ANALYSIS

Optimal Control Theory and Applications

Foundations of Dynamic Economic Analysis presents a modern and thorough exposition of the fundamental mathematical formalism used to study continuous time dynamic economic processes and to interpret dynamic economic behavior, namely, optimal control theory. The style of presentation, with its continual emphasis on the economic interpretation of mathematics and models, distinguishes it from several other excellent texts on the subject. This approach is aided dramatically by introducing the dynamic envelope theorem and the method of comparative dynamics early in the exposition. Accordingly, motivated and economically revealing proofs of the transversality conditions come about by use of the dynamic envelope theorem. Furthermore, such sequencing of the material naturally leads to the development of the primal-dual method of comparative dynamics and dynamic duality theory, two modern approaches used to tease out the empirical content of optimal control models. The stylistic approach ultimately draws attention to the empirical richness of optimal control theory, a feature missing in virtually all other textbooks of this type.

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Foreword

I've been basking in Michael Caputo's reflected glory since he was my graduate student in the 1980s, and I am pleased to do it again with the publication of this book. Michael is carrying on the tradition of exploring mathematical models not for their elegance or descriptive qualities, but for the refutable implications they generate. It was Paul Samuelson, who, in 1947, in his *Foundations of Economic Analysis*, first articulated the methodology that equilibrium conditions themselves were typically unobservable and sterile, and that meaningful theorems in economics consisted of statements that restricted the direction of change of decision variables when the data or parameters of a system changed in an observable way. In the traditional comparative statics models, such as those I analyzed in *The Structure of Economics*, refutable propositions emerged from a maximization hypothesis in a static framework. However, there has never been until now, a treatise that extended Samuelson's methodology to dynamic models, where decisions today affect the entire time path of events in the future.

Michael Caputo's contribution here and elsewhere has been to keep resource and other economists focused on this central issue of scientific methodology so that we can see the scientific usefulness of dynamic models: what refutable implications do they generate? If some initial stock or other parameter in a dynamic resource model, say, increases, under what circumstances can we make a definitive statement about the way the path or terminal conditions of the state and control variables respond? Here at last we have a systematic treatment of this class of problems that comprise positive or scientific economic analysis of dynamic models.

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Preface

Forward-looking individuals recognize that decisions made today affect those to be made in the future, at least in part, by expanding or contracting the set of admissible choices, that is, by lowering or raising the cost of a future choice. Such intertemporal linkages reside at the core of all dynamic processes in economics. Consequently, mathematical methods that account for such intertemporal linkages are fundamental, in principle, to all economic decisions. This book presents an introductory but thorough exposition of the fundamental mathematical formalism used to study continuous time dynamic economic processes and to interpret dynamic economic behavior, namely, optimal control theory and its ancillary techniques.

The style of presentation distinguishes the book from several other excellent texts on the subject. First of all, there is a continual emphasis on the economic interpretation of the mathematics and the models, aided dramatically by way of the method of comparative dynamics, from both a primal and dual point of view. In my twelve years of teaching the course upon which the book is based, I have found that this approach permits the students to garner a deeper conceptual understanding of intertemporal economic models. One of the keys to a deeper conceptual understanding, I believe, is the introduction of the dynamic envelope theorem relatively early in the presentation of optimal control theory. This strategy paves the way for many of the succeeding chapters of the book to be built around this vital theorem, and not only complements the aforementioned emphasis on economic interpretation and conceptualization, but also leads to motivated proofs of the transversality conditions, an area long on technicalities and short on intuition. Furthermore, this sequencing of the material naturally leads to the development of the primal-dual method of comparative dynamics and dynamic duality theory, two modern approaches used to tease out the empirical content of optimal control models. In sum, the stylistic approach ultimately draws attention to the empirical richness of optimal control theory, something missing in virtually all other such textbooks.

Several other novel features of the book are worth mentioning. For one, each chapter contains numerous fully worked examples, a feature of the book on which students have always commented positively. Moreover, the inclusion of several

examples in each chapter is typically the best way for newcomers to the fold to learn the material, for as is well known, learning characteristically occurs gradually, by conquering the specific before the general. The examples range from the simple mathematical variety designed to demonstrate how a particular theorem is applied to solve a control problem, to more sophisticated ones, in which a nontrivial economic problem is motivated, set up, solved either explicitly or qualitatively, and then usually scrutinized for its comparative dynamics properties. In a similar vein, the comparative dynamics methods are applied to seminal economic models rather than just simple mathematical problems, adding further to the development of the students' conceptual understanding of intertemporal economic theory. In addition, every chapter ends with a vast array of mental exercises. These range from routine and purely mathematical problems designed as a check on one's basic understanding of the material (many of which have been culled from the cited references), to proofs of theorems given in a chapter, to more complicated and detailed economic problems designed as a much deeper check on one's understanding and as an extension of the development given in the text (all of which originated in my teaching of the subject matter). There are even exercises that explore the consequences of some of the technical assumptions made in the text, for those interested in such matters.

The essential mathematics prerequisites are a standard introductory calculus sequence that includes vector calculus, a basic course in linear algebra, and an introductory course in ordinary differential equations. In other words, the archetypal two-year mathematics sequence that is taken by engineering, physics, and mathematics majors, and expected of all incoming Ph.D. students in economics, is the fundamental mathematics prerequisite. As far as training in economics is concerned, one Ph.D. level course in microeconomic theory focusing on the neoclassical theory of the consumer and the firm and their comparative static properties is essential. This is because oftentimes motivation for an intertemporal economic concept is built upon its counterpart from static microeconomic theory. The best microeconomic theory text in this regard is Silberberg's *The Structure of Economics*, as my book's focus and style is akin to his in that it is long on explanation, economic interpretation, and the development of the empirically relevant features of economic models. In fact, I have more or less used Silberberg's *The Structure of Economics* as the scaffolding for this book.

The textbook is aimed at first-year and second-year Ph.D. students in economics, agricultural and resource economics, operations research, and management science who wish a thorough but elementary treatment and economic interpretation of continuous time dynamic optimization methods and their use in economics and allied areas. More generally, the target audience I had in mind when writing the book comprises the *users* of optimal control theory, not seasoned veterans of dynamic optimization methods or individuals who are interested in developing further theorems in this area. In accord with this vision, I have adopted basic assumptions that are consistent with this audience and with the use of optimal control techniques in professional journals. Consequently, the latter two categories of individuals may

find the book long on motivation, explanation, and economic interpretation, and thus may desire a book that is more direct and terse. To them, I say go get any of the mathematics texts on the subject. Nonetheless, the statements of the definitions and theorems are comparatively rigorous by modern mathematical standards because I have taken them from mathematics texts. They are not, however, stated at the highest level of generality because of the aforementioned audience I have in mind. The prior two claims also apply to the proofs of the theorems, except in a few instances in which the proofs are altogether omitted, as in the case of generalized necessary conditions.

Based on my experiences teaching from earlier drafts of the book, there is more than enough material contained in it for a four-unit quarter-length course (10 weeks) or a three-unit semester-length course (15 weeks) in optimal control theory with economics applications. In fact, if one wishes to probe all the subjects thoroughly, it would take a year to cover them all. Nonetheless, one could easily pick and choose chapters based on one's preferences and intended course, as the book provides a good deal of flexibility. In the four-unit quarter-length course that I taught at University of California, Davis, to Ph.D. students in the Department of Agricultural and Resource Economics and the Department of Economics, I'd typically cover Chapters 1–5, 9, 10, 12, 13, 14, and one or two of Chapters 15, 16, and 17. I would assign long weekly problem sets, usually consisting of a half dozen or so exercises, as I have found that this is the best way for the students to internalize the material. The students who complete the course then characteristically take a three-unit quarter-length course in natural resource economics the following quarter, which applies the methods the students just learned. Chapters 7, 8, 11, 18, 19, and 20 are then covered in an advanced topics course in economic dynamics.

Several of my teachers and colleagues have been influential in my development as an economist and thus of this book. In chronological order, they are Randy Nelson, who first piqued my serious interest in microeconomic theory when I took intermediate price theory from him in the fall of 1981; Jim Mulligan, who, as my first mentor, regularly handed out sagacious advice and set high standards for himself, all of which still resonate with me today; Gene Silberberg, who took me to the next level by way of his first-quarter Ph.D. microeconomic theory course and who asked me a single, probing question about my research interests as a third-year Ph.D. student that has largely defined my research career; and finally Quirino Paris, who challenged and stimulated me to think deeply and more importantly, symmetrically, about economic questions. To each and every one of them, I give my heartfelt thanks and appreciation. There is no doubt that they all are responsible, in part, for the researcher and person I am today. All of my former students at UC Davis who provided me with feedback on the previous drafts of the book also deserve thanks. Specifically, I'd like to single out Lone Gronbæk, Jim Murphy, and Neill Norman for truly outstanding effort in this regard. In addition, Scott Parris, my editor at Cambridge University Press, deserves a big thank you for helping to shape the book and improve its quality. Finally, I wish to thank the four reviewers of

the text, namely, Gerhard Sorger of the University of Vienna; L. Joe Moffitt of the University of Massachusetts, Amherst; Subal Kumbhakar of SUNY Binghamton; and one who wished to remain anonymous, for their thoughtful and constructive comments. Gerhard Sorger deserves special mention, as he provided me with literally hundreds of minor comments and corrections, and dozens of deep and probing observations, essentially all of which made it into the text. All of these individuals, indubitably, helped improve the final product.