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Wavelet analysis: a new tool in physics

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Abstract

We review the general properties of the wavelet transform, both in its continuous and its discrete versions, in one or more dimensions. We also indicate some generalizations and applications in physics.

1.1 What is wavelet analysis?

Wavelet analysis is a particular time- or space-scale representation of signals which has found a wide range of applications in physics, signal processing and applied mathematics in the last few years. In order to get a feeling for it and to understand its success, let us consider first the case of one-dimensional signals.

It is a fact that most real life signals are nonstationary and usually cover a wide range of frequencies. They often contain transient components, whose apparition and disparition are physically very significant. In addition, there is frequently a direct correlation between the characteristic frequency of a given segment of the signal and the time duration of that segment. Low frequency pieces tend to last a long interval, whereas high frequencies occur in general for a short moment only. Human speech signals are typical in this respect: vowels have a relatively low mean frequency and last quite long, whereas consonants contain a wide spectrum, up to very high frequencies, especially in the attack, but they are very short.

Clearly standard Fourier analysis is inadequate for treating such signals, since it loses all information about the time localization of a given frequency component. In addition, it is very uneconomical: when the signal is almost flat, i.e. uninteresting, one still has to sum an infinite alternating series for reproducing it. Worse yet, Fourier analysis is highly unstable with respect to perturbation, because of its global character. For instance, if one adds an extra term, with a very small amplitude, to a linear superposition of sine waves, the signal will barely be modified, but the Fourier spectrum will be completely perturbed. This does not happen if the signal is represented in terms of *localized* components.

For all these reasons, signal analysts turn to *time-frequency* (TF) representations. The idea is that one needs *two* parameters: one, called a, characterizes the frequency, the other one, b, indicates the position in the signal. This concept of a TF representation is in fact quite old and familiar. The most obvious example is simply a musical score!

If one requires in addition the transform to be *linear*, a general TF transform will take the form:

$$s(x) \mapsto S(a,b) = \int_{-\infty}^{\infty} \overline{\psi_{ab}(x)}\, s(x)\, dx, \tag{1.1}$$

where s is the signal and ψ_{ab} the analysing function. Within this class, two TF transforms stand out as particularly simple and efficient: the Windowed or Short Time Fourier Transform (WFT) and the Wavelet Transform (WT). For both of them, the analysing function ψ_{ab} is obtained by acting on a basic (or mother) function ψ, in particular b is simply a time translation. The essential difference between the two is in the way the frequency parameter a is introduced.

(1) *Windowed Fourier Transform:*

$$\psi_{ab}(x) = e^{ix/a}\, \psi(x-b). \tag{1.2}$$

Here ψ is a window function and the a-dependence is a modulation ($1/a \sim$ frequency); the window has constant width, but the lower a, the larger the number of oscillations in the window (see Figure 1.1 (left))

(2) *Wavelet transform:*

$$\psi_{ab}(x) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{x-b}{a}\right). \tag{1.3}$$

The action of a on the function ψ (which must be oscillating, see below) is a dilation ($a > 1$) or a contraction ($a < 1$): the shape of the function is unchanged, it is simply spread out or squeezed (see Figure 1.1 (right)).

The WFT transform was originally introduced by Gabor (actually in a discretized version), with the window function ψ taken as a Gaussian; for this reason, it is sometimes called the Gabor transform. With this choice, the function ψ_{ab} is simply a canonical (harmonic oscillator) coherent state [17], as one sees immediately by writing $1/a = p$. Of course this book is concerned

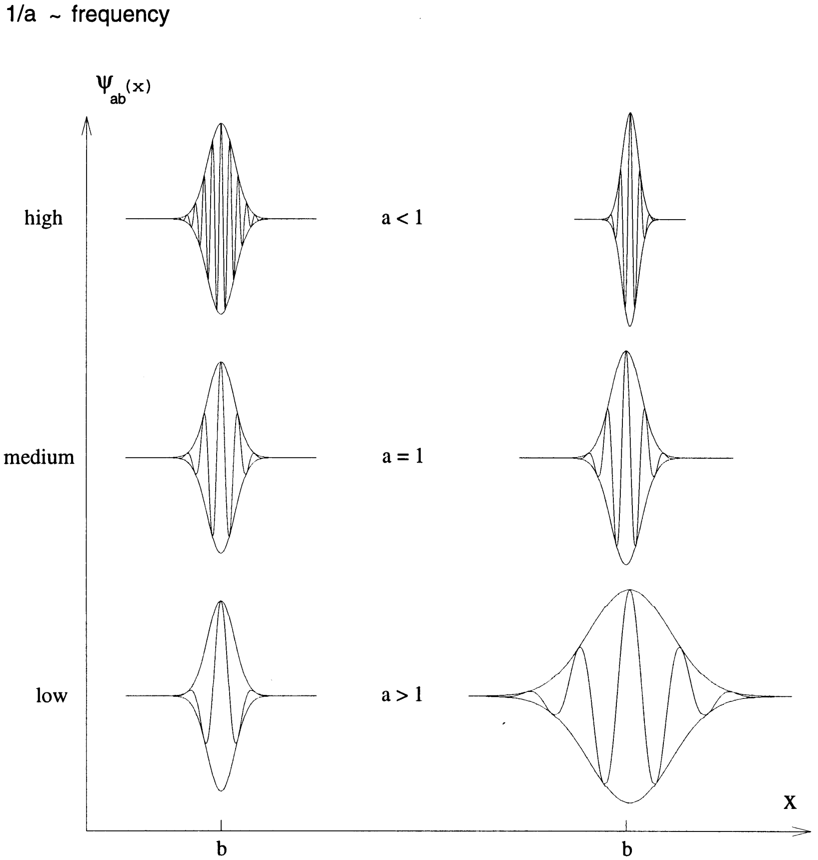

Fig. 1.1. The function $\psi_{ab}(x)$ for increasing values of $1/a \sim$ frequency, in the case of the Windowed Fourier Transform (left) and the wavelet transform (right).

essentially with the wavelet transform, but the Gabor transform will occasionally creep in, as for instance in Chapter 8.

One should note that the assumption of linearity is nontrivial, for there exists a whole class of quadratic, or more properly sesquilinear, time-frequency representations. The prototype is the so-called Wigner–Ville transform, introduced originally by E.P. Wigner in quantum mechanics (in 1932!) and extended by J. Ville to signal analysis:

$$W_s(a, b) = \int e^{-ix/a} \overline{s\left(b - \frac{x}{2}\right)} \, s\left(b + \frac{x}{2}\right) dx. \tag{1.4}$$

Further information may be found in [6, 11].

1.2 The continuous WT

Actually one should distinguish two different versions of the wavelet transform, the *continuous* WT (CWT) and the *discrete* (or more properly, discrete time) WT (DWT) [10,14]. The CWT plays the same rôle as the Fourier transform and is mostly used for analysis and feature detection in signals, whereas the DWT is the analogue of the Discrete Fourier Transform (see for instance [4] or [29]) and is more appropriate for data compression and signal reconstruction. The situation may be caricatured by saying that the CWT is more natural to the physicist, while the DWT is more congenial to the signal analyst and the numericist. This explains why the CWT will play a major part in this book.

The two versions of the WT are based on the same transformation formula, which reads, from (1.1) and (1.3):

$$S(a,b) = a^{-1/2} \int_{-\infty}^{\infty} \overline{\psi\left(\frac{x-b}{a}\right)}\, s(x)\, dx, \tag{1.5}$$

where $a > 0$ is a scale parameter and $b \in \mathbb{R}$ a translation parameter. Equivalently, in terms of Fourier transforms:

$$S(a,b) = a^{1/2} \int_{-\infty}^{\infty} \overline{\widehat{\psi}(a\omega)}\, \widehat{s}(\omega) e^{ib\omega}\, d\omega. \tag{1.6}$$

In these relations, s is a square integrable function (signal analysts would say: a finite energy signal) and the function ψ, the analysing wavelet, is assumed to be well localized *both* in the space (or time) domain and in the frequency domain. In addition ψ must satisfy the following admissibility condition, which guarantees the invertibility of the WT:

$$\int_{-\infty}^{\infty} |\widehat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} < \infty. \tag{1.7}$$

In most cases, this condition may be reduced to the requirement that ψ has zero mean (hence it must be oscillating):

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0. \tag{1.8}$$

In addition, ψ is often required to have a certain number of *vanishing moments:*

$$\int_{-\infty}^{\infty} x^n\, \psi(x)\, dx = 0, \;\; n = 0, 1, \ldots, N. \tag{1.9}$$

This property improves the efficiency of ψ at detecting singularities in the signal, since it is blind to polynomials up to order N.

One should emphasize here that the choice of the normalization factor $a^{-1/2}$ in (1.3) or (1.5) is not essential. Actually, one often uses instead a factor a^{-1} (the so-called L^1 normalization), and this has the advantage of giving more weight to the small scales, i.e. the high frequency part (which contains the singularities of the signal, if any). The choice $a^{-1/2}$ makes the transform unitary: $\|\psi_{ab}\| = \|\psi\|$ and also $\|S\| = \|s\|$, where $\|\cdot\|$ denotes the L^2 norm in the appropriate variables (the squared norm is interpreted as the total energy of the signal).

Notice that, instead of (1.5), which defines the WT as the scalar product of the signal s with the transformed wavelet ψ_{ab}, $S(a, b)$ may also be seen as the convolution of s with the scaled, flipped and conjugated wavelet $\widetilde{\psi}_a(x) = a^{-1/2}\,\overline{\psi(-x/a)}$:

$$S(a, b) = (\widetilde{\psi}_a * s)(b) = \int_{-\infty}^{\infty} \widetilde{\psi}_a(b - x)\, s(x)\, dx. \tag{1.10}$$

In other words, the CWT acts as a *filter* with a function of zero mean.

This property is crucial, for the main virtues of the CWT follow from it, combined with the support properties of ψ. Indeed, if we assume ψ and $\widehat{\psi}$ to be as well localized as possible (but respecting the Fourier uncertainty principle), then so are the transformed wavelets ψ_{ab} and $\widehat{\psi}_{ab}$. Therefore, the WT $s \mapsto S$ performs a *local filtering*, both in time (b) and in scale (a). The transform $S(a, b)$ is nonnegligible only when the wavelet ψ_{ab} matches the signal, that is, the WT selects the part of the signal, if any, that lives around the time b and the scale a.

In addition, if $\widehat{\psi}$ has an essential support (bandwidth) of width Ω, then $\widehat{\psi}_{ab}$ has an essential support of width Ω/a. Thus, remembering that $1/a$ behaves like a frequency, we conclude that the WT works at constant *relative* bandwidth, that is, $\Delta\omega/\omega =$ constant. This implies that it is very efficient at high frequency, i.e. small scales, in particular for the detection of singularities in the signal. By comparison, in the case of the Gabor transform, the support of $\widehat{\psi}_{ab}$ keeps the same width Ω for all a, that is, the WFT works at constant bandwidth, $\Delta\omega =$ constant. This difference in behaviour is often the key factor in deciding whether one should choose the WFT or the WT in a given physical problem (see for instance Chapter 8).

Another crucial fact is that the transformation $s(x) \mapsto S(a, b)$ may be inverted exactly, which yields a reconstruction formula (this is only the simplest one, others are possible, for instance using different wavelets for the decomposition and the reconstruction):

$$s(x) = c_\psi^{-1} \int_{-\infty}^{\infty} db \int_0^{\infty} \frac{da}{a^2}\ \psi_{ab}(x) S(a,b), \tag{1.11}$$

where c_ψ is a normalization constant. This means that the WT provides a decomposition of the signal as a linear superposition of the wavelets ψ_{ab} with coefficients $S(a, b)$. Notice that the natural measure on the parameter space (a, b) is $da\, db/a^2$, and it is invariant not only under time translation, but also under dilation. This fact is important, for it suggests that these geometric transformations play an essential rôle in the CWT. This aspect will be discussed thoroughly in Chapter 2.

All this concerns the continuous WT (CWT). But, in practice, for numerical purposes, the transform must be *discretized*, by restricting the parameters a and b in (1.5) to the points of a lattice, typically a dyadic one:

$$S_{j,k} = 2^{-j/2} \int_{-\infty}^{\infty} \overline{\psi(2^{-j}x - k)}\, s(x)\, dx, \quad j, k \in \mathbb{Z}. \tag{1.12}$$

Then the reconstruction formula (1.11) becomes simply

$$s(x) = \sum_{j,k \in Z} S_{j,k}\ \widetilde{\psi_{j,k}}(x), \tag{1.13}$$

where the function $\widetilde{\psi_{j,k}}$ may be explicitly constructed from $\psi_{j,k}$. In this way, one arrives at the theory of *frames* or nonorthogonal expansions [9, 10], which offer a good substitute to orthonormal bases. Very general functions ψ satisfying the admissibility condition (1.7) described above will yield a good frame, but not an orthonormal basis, since the functions $\{\psi_{j,k}(x) \equiv 2^{j/2}\psi(2^j x - k), j, k \in \mathbb{Z}\}$ are in general not orthogonal to each other!

Yet orthonormal bases of wavelets can be constructed, but by a totally different approach, based on the concept of *multiresolution analysis*. We emphasize that the discretized version of the CWT just described is totally different in spirit and method from the genuine DWT, to which we now turn. The full story may be found in [10], for instance.

1.3 The discrete WT: orthonormal bases of wavelets

One of the successes of the WT was the discovery that it is possible to construct functions ψ for which $\{\psi_{j,k}, j, k \in \mathbb{Z}\}$ is indeed an orthonormal basis of $L^2(\mathbb{R})$.

In addition, such a basis still has the good properties of wavelets, including space *and* frequency localization. Moreover, it yields fast algorithms, and this is the key to the usefulness of wavelets in many applications

The construction is based on two facts: first, almost all examples of orthonormal bases of wavelets can be derived from a multiresolution analysis, and then the whole construction may be transcribed into the language of digital filters, familiar in the signal processing literature.

A *multiresolution analysis* of $L^2(\mathbb{R})$ is an increasing sequence of closed subspaces

$$\ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots, \tag{1.14}$$

with $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j \in \mathbb{Z}} V_j$ dense in $L^2(\mathbb{R})$ (loosely speaking, this means $\lim_{j\to\infty} V_j = L^2(\mathbb{R})$), and such that

(1) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$
(2) there exists a function $\phi \in V_0$, called a *scaling function*, such that the family $\{\phi(x-k), k \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

Combining conditions (1) and (2), one gets an orthonormal basis of V_j, namely $\{\phi_{j,k}(x) \equiv 2^{j/2}\phi(2^j x - k), k \in \mathbb{Z}\}$. Note that we may take for ϕ a *real* function, since we are dealing with signals.

Each V_j can be interpreted as an *approximation space*: the approximation of $f \in L^2(\mathbb{R})$ at the resolution 2^{-j} is defined by its projection onto V_j, and the larger j, the finer the resolution obtained. Then condition (1) means that no scale is privileged. The additional details needed for increasing the resolution from 2^{-j} to $2^{-(j+1)}$ are given by the projection of f onto the orthogonal complement W_j of V_j in V_{j+1}:

$$V_j \oplus W_j = V_{j+1}, \tag{1.15}$$

and we have:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j. \tag{1.16}$$

Equivalently, fixing some lowest resolution level j_o, one may write

$$L^2(\mathbb{R}) = V_{j_o} \oplus \left(\bigoplus_{j \geq j_o} W_j \right). \tag{1.17}$$

Then the theory asserts the existence of a function ψ, called the *mother wavelet*, explicitly computable from ϕ, such that $\{\psi_{j,k}(x) \equiv 2^{j/2}\psi(2^j x - k), j, k \in \mathbb{Z}\}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$: these are the *orthonormal wavelets*.

The construction of ψ proceeds as follows. First, the inclusion $V_0 \subset V_1$ yields the relation (called the scaling or refining equation):

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_n \phi(2x - n), \quad h_n = \langle \phi_{1,n} | \phi \rangle. \tag{1.18}$$

Taking Fourier transforms, this gives

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2), \tag{1.19}$$

where

$$m_0(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} h_n e^{-in\omega} \tag{1.20}$$

is a 2π-periodic function. Iterating (1.19), one gets the scaling function as the (convergent!) infinite product

$$\widehat{\phi}(\omega) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\omega). \tag{1.21}$$

Then one defines the function $\psi \in W_0 \subset V_1$ by the relation

$$\widehat{\psi}(\omega) = e^{i\omega/2}\, \overline{m_0(\omega/2 + \pi)}\, \widehat{\phi}(\omega/2), \tag{1.22}$$

or, equivalently

$$\psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^{n-1} h_{-n-1} \phi(2x - n), \tag{1.23}$$

and proves that the function ψ indeed generates an orthonormal basis with all the required properties.

Various additional conditions may be imposed on the function ψ (hence on the basis wavelets): arbitrary regularity, several vanishing moments (in any case, ψ has always mean zero), symmetry, fast decrease at infinity, even compact support. The technique consists in translating the multiresolution structure into the language of digital filters. Actually this means nothing more than expanding (filter) functions in a Fourier series. For instance, (1.19) means that $m_0(\omega)$ is a filter (multiplication operator in frequency space), with filter coefficients h_n. Similarly, (1.22) may be written in terms of the filter $m_1(\omega) = e^{i\omega}\, \overline{m_0(\omega + \pi)}$. (Notice that this particular relation between m_0, m_1, together with the identity $|m_0(\omega)|^2 + |m_1(\omega)|^2 = 1$, define what electrical engineers call a Quadrature Mirror Filter or QMF.) Then the various restrictions imposed on ψ translate into suitable constraints on the filter coefficients h_n. For instance, ψ has compact support if only finitely many h_n differ from zero.

The simplest example of this construction is the Haar basis, which comes from the scaling function $\phi(x) = 1$ for $0 \leq x < 1$ and 0 otherwise. Similarly, various spline bases may be obtained along the same line. Other explicit examples may be found in [5] or [10].

In practical applications, the (sampled) signal is taken in some V_J, and then the decomposition (1.17) is replaced by the finite representation

$$V_J = V_{j_o} \oplus \left(\bigoplus_{j=j_o}^{J-1} W_j \right). \tag{1.24}$$

Figure 1.2 shows an example (obtained with the MATLAB Wavelet Toolbox [3]) of a decomposition of order 5, namely

$$V_0 = V_{-5} \oplus W_{-5} \oplus W_{-4} \oplus W_{-3} \oplus W_{-2} \oplus W_{-1}. \tag{1.25}$$

As we just saw, appropriate filters generate orthonormal wavelet bases. However, this result turns out to be too rigid and various generalizations have been proposed (see [25] for details).

(i) *Biorthogonal wavelet bases*:
As we mentioned in Section 1.2, the wavelet used in the CWT for reconstruction need not be the same as that used for decomposition, the two have only to satisfy a cross-compatibility condition. The same idea in the discrete case leads to biorthogonal bases, i.e. one has two hierarchies of approximation spaces, V_j and $\check{V}_j$, with cross-orthogonality relations. This gives a better control, for instance, on the regularity or decrease properties of the wavelets.

(ii) *Wavelet packets and the best basis algorithm*:
The construction of orthonormal wavelet bases leads to a special subband coding scheme, rather asymmetrical: each approximation space V_j gets further decomposed into V_{j-1} and W_{j-1}, whereas the detail space W_j is left unmodified. Thus more flexible subband schemes have been considered, called *wavelet packets*; they provide rich libraries of orthonormal bases, and also strategies for determining the optimal basis in a given situation [7, 32].

(iii) *The lifting scheme*:
One can go one step beyond, and abandon the regular dyadic scheme and the Fourier transform altogether. The resulting method leads to the so-called *second-generation wavelets* [31], which are essentially custom-designed for any given problem.

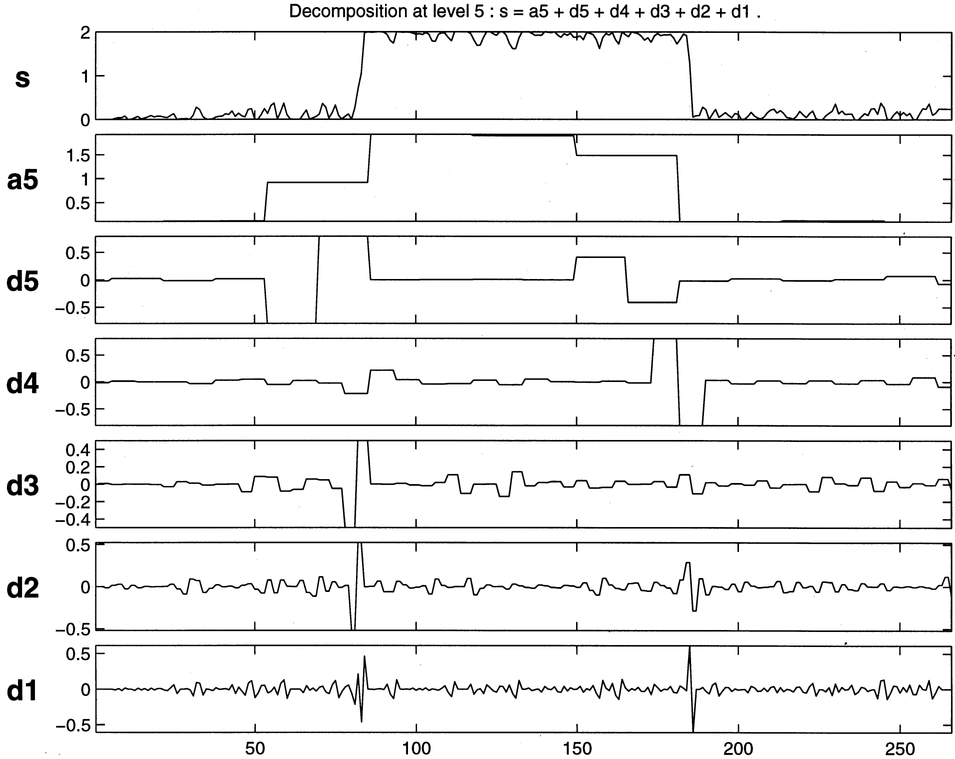

Fig. 1.2. A decomposition of order 5. The signal s lives in V_0 and it is decomposed into its approximation $a_5 \in V_{-5}$ and the increasingly finer details $d_j \in W_{-j}$, $j = 5, 4, 3, 2, 1$.

1.4 The wavelet transform in more than one dimension

Wavelet analysis may be extended to 2-D signals, that is, in *image analysis*. This extension was pioneered by Mallat [19, 20], who developed systematically a 2-D *discrete* (but redundant) WT. This generalization is indeed a very natural one, if one realizes that the whole idea of multiresolution analysis lies at the heart of human vision. In fact, most of the concepts are indeed already present in the pioneering work of Marr [22] on vision modelling. As in 1-D, this discrete WT has a close relationship with numerical filters and related techniques of signal analysis, such as subband coding. It has been applied successfully to several standard problems of image processing. As a matter of fact, all the approaches that we have mentioned above in the 1-D case have been extended to 2-D: orthonormal bases, biorthogonal bases, wavelet packets, lifting scheme. These topics will be discussed in detail in Chapter 2.

However, the continuous transform may also be extended to 2 (or more) dimensions, with exactly the same properties as in the 1-D case [2, 26]. Here again the mechanism of the WT is easily understood from its very definition as a convolution (in the sense of (1.10)):

$$S(a, \theta, \vec{b}) \sim \int d^2\vec{x}\ \overline{\bar{\psi}(a^{-1} r_{-\theta}(\vec{x} - \vec{b}))s(\vec{x})}, \quad a > 0, 0 \leq \theta < 2\pi, b \in \mathbb{R}^2, \tag{1.26}$$

where s is the signal and ψ is the analysing wavelet, which is translated by $\vec{b}$, dilated by a and rotated by an angle θ ($r_{-\theta}$ is the rotation operator). Since the wavelet ψ is required to have zero mean, we have again a filtering effect, i.e. the analysis is *local* in all four parameters $a, \theta, \vec{b}$, and here too it is particularly efficient at detecting discontinuities in images.

Surprisingly, most applications have treated the 2-D WT as a 'mathematical microscope', like in 1-D, thus ignoring directions. This is particularly true for the discrete version. There, indeed, a 2-D multiresolution is simply the tensor product of two 1-D schemes, one for the horizontal direction and one for the vertical direction (in technical terms, one uses only separable filters). However the 2-D *continuous* WT, including the orientation parameter θ, may be used for detecting *oriented features* of the signal, that is, regions where the amplitude is regular along one direction and has a sharp variation along the perpendicular direction, for instance, in the classical problem of *edge detection*. The CWT is a very efficient tool in this respect, provided one uses a wavelet which has itself an intrinsic orientation (for instance, it contains a plane wave). For this reason, a large part of Chapter 2 will be devoted to the continuous WT and its applications.

For further extensions of the CWT, it is crucial to note that the 2-D version comes directly from group representation theory, the group in this case being the so-called similitude group of the plane, consisting of translations, rotations and global dilations [26]. Note that the 1-D CWT may also be derived from group theory [10], in that case from the so-called '$ax + b$' group of dilations and translations of the line.

What we have here is in fact a general pattern. Consider the class of finite energy signals living on a manifold Y, i.e. $s \in L^2(Y, d\mu) \equiv \mathcal{H}$. For instance, Y could be space $\mathbb{R}^n$, the 2-sphere S^2, space-time $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2 \times \mathbb{R}$, etc. Suppose there is a group G of transformations acting on Y, that contains *dilations* of some kind. As usual, this action will be expressed by a unitary representation U of G in the space $\mathcal{H}$ of signals. Then, under a simple technical assumption on U ('square integrability'), a wavelet analysis on Y, adapted to the symmetry group G, may be constructed, following the general construction of coherent states on Y associated to G [1]. This technique has been implemented successfully for extending the CWT to higher dimensions (in 3-D, for instance, one gets a tool for target tracking), the 2-sphere (a tool most wanted by geophysicists) or to space-time (time-dependent signals or images, such as TV or video sequences), including relativistic effects (using wavelets associated to the affine Galilei or Poincaré group). This general approach will be described with all the necessary mathematical details in Chapter 2.

It is interesting to remark that the CWT was in fact designed by physicists. The idea of deriving it from group theory is entirely natural in the framework of coherent states [1, 17], and the connection was made explicitly from the very beginning [12, 13]. In a sense, the CWT consists in the application of ideas from quantum physics to signal and image processing. The resulting effect of cross-fertilization may be one of the reasons of its richness and its success.

1.5 Outcome

As a general conclusion, it is fair to say that the wavelet techniques have become an established tool in signal and image processing, both in their CWT and DWT incarnations and their generalizations. They are being incorporated as a new tool in many reference books and software codes. They have distinct advantages over concurrent methods by their adaptive character, manifested for instance in their good performances in pattern recognition or directional filtering (in the case of the CWT), and by their very economical aspect, achieved in impressive compression rates (in the case of the DWT). This is especially useful in image processing, where huge amount of data, mostly redundant, have to be stored and transmitted.

As a consequence, they have found applications in many branches of physics, such as acoustics, spectroscopy, geophysics, astrophysics, fluid mechanics (turbulence), medical imagery, atomic physics (laser–atom interaction), solid state physics (structure calculations), ... Some of these results will be reviewed in the subsequent chapters. For additional information, see [24].

Thus we may safely bet that wavelets are here to stay, and that they have a bright future. Of course wavelets don't solve every difficulty, and must be continually developed and enriched, as has been the case over the last few years. In particular, one should expect a proliferation of specialized wavelets, each dedicated to a particular type of problem, and an increasingly diverse spectrum of physical applications. This trend is only natural, it follows from the very structure of the wavelet transform – and in that respect the wavelet philosophy is exactly opposite to that of the Fourier transform, which is usually seen as a universal tool.

Finally a word about references. The literature on wavelet analysis is growing exponentially, so that some guidance may be helpful. As a first contact, an introductory article such as [29] may be a good suggestion, followed by the the popular, but highly successful book of Burke Hubbard [4]. Slightly more technical, but still elementary and aimed at a wide audience, are the books of Meyer [25] and Ogden [27]. While the former is a nice introduction to the mathematical ideas underlying wavelets, the latter focuses more on the statistical aspects of data analysis. Note that, since wavelets have found applications in most branches of physics, pedestrian introductions on them have been written in the specialized journals of each community (to give an example, meteorologists will appreciate [18]).

For a survey of the various applications, and a good glimpse of the chronological evolution, there is still no better place to look than the proceedings of the three large wavelet conferences, Marseille 1987 [8], Marseille 1989 [23] and Toulouse 1992 [24]. Finally a systematic study requires a textbook. Among the increasing number of books and special issues of journals appearing on the market, we recommend in particular the volumes of Daubechies [10], Chui [5], Kaiser [16] and Holschneider [14], the collection of review articles in [30] and several special issues of IEEE journals [15,28]. In particular, [3] gives a useful survey of the available software related to wavelets. Another good choice, complete but accessible to a broad readership, is the recent textbook of Mallat [21].

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