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A guided tour through the book

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The reader might want to jump right into the book, but I decided to give a guided tour (which one may leave and rejoin at will of course) through the chapters, to whet the reader's taste.

Antoine opens in Chapter 1 with a brief survey of the basic properties of wavelet transforms, both continuous (CWT) and discrete (DWT). In the latter case one learns about the intuitively very appealing concept of *multiresolution analysis*. Section 1.4 looks ahead to the two- and more-dimensional versions, and summarily brings out connections with well known symmetry groups of physics, and the theory of coherent states.

In the second chapter, also by Antoine, the 2-D wavelet transform is treated. Here the characterization as mathematical microscope must be further qualified, because it misses the new and important property of *orientability* of the 2-D wavelets, which the 1-D case lacks. A real-world microscope is not more sensitive in one direction than in another one, it is 'isotropic'. But the mathematical microscope as embodied in 2-D wavelets has an extra feature: these wavelets can be designed in such a way that they are *directionally selective*. Apart from dilation and translation, one can now also *rotate* the wavelet, which makes possible a sensitive detection of oriented features of a signal (a 2-D image). In many texts the 2-D case is still limited to the DWT, and the wavelets are usually formed by taking tensor products of 1-D wavelets in the x and y-direction, thereby giving preference to horizontal, vertical and diagonal features in the plane. The continuous case is described here in some detail, first because it admits interesting physical applications, such as measuring the velocity field of a 2-D turbulent flow around an obstacle, the disentangling of a superposition of damped plane waves under water produced by a source above the water surface, fault detection in geology, analysis of spectra, contrast enhancement of images. By using the *scale-angle measure* one can exhibit symmetries of objects. Another neat example under development is the detection of Einstein rings by using an annular-shaped wavelet at a fixed scale, leading to, e.g., distance measurements of quasars. The second reason to devote much attention to the 2-D CWT is that the mathematical background, as mapped out in section 2.4, brings out the connections with group representations and coherent states, both used in physics long before wavelets came into the picture. It turns out that wavelets are the coherent states associated to the similitude groups (Euclidean groups with dilations). This section is mathematically somewhat more abstract than the rest of the chapter. The importance of it is that it is shown here, how one can extend the CWT to other 'spaces', such as 3-D space, the sphere, and to space-time ('kinematical' wavelets used in motion tracking, including relativistic effects (using wavelets associated to the Galilei or Poincaré group resp.). Also some applications of the 2-D DWT are indicated.

In Chapter 3 we turn to applications on the largest scale in the Universe: Bijaoui describes a wide variety of applications in astrophysics and observational cosmology. The wavelet transform is a very good tool to study power-law signals, and these occur in many astrophysical sources, such as the light intensity of the solar surface, the brightness of interstellar clouds, or galaxy distributions from galaxy counts. Often the power law behaviour is exhibited by statistical correlation functions, so in many applications there is a combination of statistical techniques with wavelet methods. Cluster analysis of galaxies for instance, was much improved. Image compression is frequently needed in astronomy. Much work was done on Hubble Space Telescope (HST) images and astronomical aperture synthesis. The DWT is not only used in the form resulting from multiresolution analysis, but also by other methods: the 'à trous algorithm', and the 'pyramidal transform' are used for image restoration and analysis. Denoising images also receives a good deal of attention: criteria to establish the notion of 'significant coefficient' were developed. Connected with that is the problem of deconvolving an observed signal (image) to obtain the true object signal: that is the signal before it is convolved with the response function (called the 'point spread function' in optics) of the measuring apparatus. Multiresolution techniques yield a good reduction of resolution here, especially for HST data. To obtain an automated image analysis for astronomical images, one needs a so called 'vision model': a protocol of operations to analyse the image. The classical examples of this were based on edge detection, but this is not adequate to recognize astronomical objects accurately. In a typical image one can see point-like sources (stars), quasi-point-like objects (double stars, faint galaxies...) and complex diffuse structures (galaxies, nebulae, clusters...). The multiscale vision model developed here is able to optimize the detection of objects, because it yields a background mapping adapted to a given object. The earlier methods were only suited to stars of quasi-stellar sources with a slowly varying background. Since in the multiscale approach the notion of subobject is defined, much more complex structures can be analysed.

From astronomical scales down to microscopically small scales one finds turbulence, naturally occurring or man-made. The study of fully developed (high Reynolds number) turbulence by means of the wavelet transform is presented by Farge *et al*. in Chapter 4. The authors argue that part of the reason the subject has not undergone fundamental progress for a long time is that point measurements are used to compute averages in the statistical theory, and also because one keeps thinking in terms of Fourier modes. Thus the presence of coherent structures (here defined as local condensations of the vorticity field which survive much longer than the typical eddy turnover time) is missed, although these are observed in physical space, and their role seems essential in the dynamics. The classical theory of turbulence is not able to see the coherent structures, because they are only felt in the high order statistical moments of the velocity increments in the flow, which have been measured only relatively recently and turn out not to obey Kolmogorov's theory. Wavelets can play a role in separating the coherent components from the incoherent components of turbulent flows, so that one can arrive at new conditional averages, replacing the classical ensemble averages. Fourier space analysis is not capable of this disentanglement, because it averages over space and thus loses local information. The coherent structures correspond to spatio-temporally quasi-singular structures, and thus the use of wavelets to analyse isolated or dense distributions of singularities is brought out, a subject that will be dealt with in extenso in Arneodo *et al*.'s Chapter 9. The separation of coherent structures and random background flow allows new proposals in the modelling of turbulence in which one may expect to be able to explore back and forth transfers of energy between coherent components and the background of the flow. Similar transfers are estimated from real-world global atmospheric data (albeit outside the turbulent regime) by Fournier in Chapter 7. Also in stochastic models of turbulence wavelets are beginning to be used.

Wavelet bases are also increasingly being used to solve partial differential equations numerically. Section 4.6 describes some examples in the literature and presents in some detail algorithms to solve the two-dimensional Navier–Stokes equations.

Coherent structures are also the subject of Chapter 5 by Hudgins and Kaspersen. They focus on the case of cylinder wake flow, and compare the performance of conventional as well as wavelet-based coherent structure detector algorithms. This performance is measured by two statistics: the probability of detection P_D, and the false alarm-rate P_{FA}, that is the probability that a detection will be reported when the relevant event is in fact not present. These quantities are dependent, and this dependence can be parametrized, giving rise to a plot of P_D vs. P_{FA}. The authors test their algorithms on a particular kind of coherent structure called a *burst*: an outrush from the wall, during which the transverse velocity is positive while the streamwise velocity temporarily falls below its mean value. Three conventional detectors are described, and two different wavelet detectors are introduced. Comparison of the results then shows that wavelet methods perform better than the conventional ones, and for high detection rates the second wavelet method outperforms all of the others.

Van Milligen aims at getting a grip on the non-linearity aspect of turbulence in Chapter 6. He defines the notions of *bispectrum* and *bicoherence* based on wavelets. The bicoherence is a measure of the amount of phase coupling that occurs in a signal or between two signals, which means that if two frequencies are simultaneously present in the signal(s) along with their sum (or difference), the sum of the phases of these frequency components is constant in time. Since the wavelet version of these notions is based on integration over a short time interval, temporal variations in phase coupling (intermittent behaviour) can be revealed. Two possible interpretations of the bicoherence are presented: one in terms of coherent structures passing by the observation point, and another one in terms of a coupling constant in a quadratic wave-interaction model. The usefulness of these concepts is first demonstrated in numerical examples: two coupled van der Pol oscillators exhibiting chaos, and then two models for plasma turbulence. It turns out that one can perform detailed spectral analysis on turbulence simulations although only short data series are available (due to CPU-time limitations) rendering Fourier analysis impracticable or impossible. In the last section van Milligen analyses in detail data from torsatron and tokamak plasma experiments.

Turning away from turbulence, in Chapter 7 we find an application, by Fournier, of wavelets to an anomalous state of the earth's atmosphere, namely *blocking*. This is a period of time during which the normal progression (approximately eastward translation) of weather patterns is locally inhibited. It is associated with a quasi-persistent anomalous high pressure system. Fournier reviews the equations derived by Saltzman for the evolution of the mean kinetic energy of eddies. The contributions to this from atmospheric structures of distinct scales are conventionally resolved by (truncated) Fourier series representations. This is replaced here by an analysis in terms of a periodic orthonormal wavelet basis. In terms of these it is possible to construct scale dependent transfer and flux functions of kinetic energy at a certain location. These new concepts are then applied to real-world data from the National Meteorological Center: wind components u (eastward) and v (northward) and the 'geopotential height' Z. Analysis of these data tells us that blocking is largely described by the largest scale part of the multiresolution analysis, and new support is found for the hypothesis that blocking is partially maintained by a particular kind of *inverse* energy cascade (going from smaller to larger scales).

Scaling down to very small distances brings us to applications of wavelets in the domains of atomic and solid state physics. In Chapter 8, Antoine *et al.* start with the case of the generation of light emission resulting from the exposure of atoms to a strong laser pulse. Odd harmonics of the laser frequency are emitted, and in order to understand the mechanism of emission better one would like to know for instance when the harmonics are emitted during the optical cycle, and what the time evolution is during the laser pulse. Standard spectral analysis cannot answer these questions. For atomic hydrogen the emission is investigated by both the Gabor Transform (Windowed Fourier Transform) and the Wavelet Transform, yielding time profiles of each individual harmonic. Analysis of these profiles leads to the conclusion that harmonic emission takes place only when the electron is close to the nucleus. The authors emphasize that in this type of analysis the Gabor transform and the wavelet transform are not each others competitors, but rather they supply complementary information, depending on the exact physical problem one studies. A further development on the basis of these results may be the temporal control of the harmonic emissions by tuning the polarization of the laser, eventually allowing the production of intense attosecond (10^{-18} s) pulses. For the case of multi-electronic wave functions orthogonal wavelet bases on $(0, \infty)$ are being proposed as a basis for the radial part of the wave function, allowing improvements over more conventional Hartree–Fock methods. A combination of wavelet transforms and conventional techniques also allowed a better calculation of energy levels in atoms.

In the second part of the chapter Antoine *et al.* deal with electronic structure calculations in solid state physics. Here both non-orthogonal and orthogonal wavelet bases have been applied successfully, and the recently developed second generation wavelets, used in a biorthogonal basis (see Ch. 1) have been used to solve a 3-D atomic Coulomb problem, namely the Poisson equation for the potential of, for instance, a uranium dimer. The potential is obtained with 6 significant digits throughout the region of interest. In the last section of this chapter the use of 2-D wavelet bases to a 2-D phenomenon, the Fractional Quantum Hall Effect, is explored.

The last three chapters all deal with phenomena in which scaling is of central importance.

Arneodo *et al*. show in Chapter 9 how wavelets can be applied to analyse the scaling properties of multifractal signals which have densely packed singularities of varying strengths. When a signal possesses a single isolated singularity at x_0, with strength, $\alpha(x_0)$ (mathematicians call this the Hölder exponent), this property is reflected in the behaviour of the wavelet transform at that location, and $\alpha(x_0)$ can therefore be extracted from, a log-log plot of the wavelet transform amplitude versus the scale. The dense packing of singularities in a multifractal signal makes straightforward application of this impossible. In order to analyse multifractal signals, a method not involving wavelets, called the *thermodynamical formalism* was developed more than a decade ago. It enables one to calculate the spectrum of singularities, the $f(\alpha)$ spectrum, by statistical means. Before the advent of wavelets this spectrum could be determined for *singular measures* only, but as the authors show, by using wavelets one can extend this to *singular functions* as well, thereby making the method applicable to any experimental signal.

Roughly speaking a (multi)fractal function is non-smooth in all or a large part of its domain, thus making traditional analytical (calculus) methods inadequate to analyse it. Unfolding the function in the wavelet domain restores the applicability of these methods. In particular, the *wavelet transform modulus maxima* (WTMM) are used to obtain a *skeleton* of the function, which provides a partition allowing the merging of the WTMM method with the thermodynamical formalism, so that the singularity spectrum can be determined. This remedies some defects of classical ‘box counting’ for measures, and of the ‘structure function’ method used for turbulent signals.

In a further development the WTMM skeleton method is used to address the ‘inverse fractal problem’: if a fractal object is produced by a dynamical system, can one then extract enough information from the object to recover the dynamical system that produced it? This is a big problem if stated in such generality, but as the authors show, one can solve this for instance in the case when the dynamics is generated by ‘cookie-cutter maps’.

Finally the method is applied to the analysis of diffusion-limited-aggregation (DLA) processes, and it is shown how one uncovers the ‘Fibonacci multiplicative process’ responsible for the branching morphology of the clusters formed by DLA. This is a remarkable result, given the geometrically featureless random walk process that generates the clusters.

The method as described in this chapter is being further developed, as Farge *et al.* mention in Ch. 4, and has also been applied for instance to the analysis of DNA nucleotide sequences (Ch. 10, ref. 81).

The application of wavelets in medicine and biology has proliferated in many different directions, witness the reference list to Chapter 10, by Ivanov *et al*. One area is the study of physiological time series, which generally have a non-stationary character. The specific case analysed here is the comparison between time series of heart beat intervals in healthy human individuals, and in patients suffering from sleep apnea. The authors develop the *cumulative variation amplitude analysis* (CVAA), consisting of a sequential application of the wavelet transform and the Hilbert transform. The first step is to take analysing wavelets that are able to eliminate the influence of linear and low-degree polynomial trends in a signal $s(t)$ (the derivatives of the Gaussian supply wavelets that can do this), keeping only in sight the variations of patterns of a certain duration a of interest. By fixing the scale parameter of the wavelet transform one obtains again a 1-D signal, say $s_a(t)$, expressing how strongly patterns with a certain duration around the value a are present within the signal. It is the variations in this strength which are of interest. The Hilbert Transform, applied to the signal $s_a(t)$ enables one to calculate an 'instantaneous amplitude' of that signal, which is an envelope of it. By counting how often in $s_a(t)$ a given instantaneous amplitude occurs, one obtains a distribution of instantaneous amplitude values which tells one what the relative length (total duration in the entire signal) of an 'a-scale pattern' with a given amplitude is. Every individual has its own amplitude distribution, but it turns out that they are scaling copies of a common distribution, at least in groups of healthy patients. Thus by rescaling individual 'healthy distributions', one can collapse them on their common distribution. Moreover this collapse repeats itself, in healthy individuals, for many different values of a. The collapse fails, however, in groups of subjects suffering from sleep apnea. These two groups can thereby be distinguished from one another. (Applying the Hilbert Transform directly to $s(t)$ itself fails to bring this out.) The authors describe how one may attempt to develop this result further into a tool to separate healthy from abnormal cardiac dynamics for an individual, thus setting up a diagnostic. Finally the relation of the scaling property with the non-linear dynamics of the heartbeat control mechanism is discussed.

The last chapter, by Guérin and Holschneider, concerns the description of intermittency in the time evolution of a system. They define the concept of a *lacunarity dimension* which quantifies the notion of intermittent behaviour. This is the only chapter where detailed mathematical proofs are presented, but we have relegated them to the Appendix so that the flow of the argument is not interrupted. Intermittency is a concept that has been mentioned many times already in previous chapters, but only qualitatively. If you think of a particle recurring intermittently in a region A of phase space, its presence in A can be described by a function $h(t) = \chi_A(x(t))$, where $x(t)$ is the trajectory in phase space, and χ_A is the characteristic function of A, registering whether or not $x(t)$ is in the region A. If one knows the dynamics $x(t)$ over a time interval $[0, T]$, one can calculate the fraction of T the particle spends in A, by taking the time average of $h(t)$ over this interval. If this fraction converges to a finite constant as $T \to \infty$, this limit can be interpreted as a rate of presence in region A. By considering not just the average of $h(t)$, but also its higher moments, the authors find the definition of the lacunarity dimension. So far, no wavelets. This definition is then applied to the case of time evolution of a system obeying the Schrödinger equation with a time independent Hamiltonian. The function $h(t)$ is now the probability to find the system in a certain region of space. The lacunarity dimension can be calculated if $h(t)$ is known over a very large time span, but this may be too long for measurements. It turns out that one can circumvent this by using wavelets to define the *generalized wavelet dimensions* of the Hamiltonian's spectral measure. The latter can be determined from time independent data which are known about the system. The main theorem of the chapter establishes that the lacunarity dimension of the time evolution generated by the Schrödinger equation is obtainable from the generalized wavelet dimensions of the spectral measure of the Hamiltonian. Thus the long time chaotic behaviour of the system and small scale spectral properties of the Hamiltonian are strictly related.

This ends our guided tour. I hope it has aroused your curiosity enough to take a closer look into the chapters that follow.

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Wavelet analysis: a new tool in physics

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Abstract

We review the general properties of the wavelet transform, both in its continuous and its discrete versions, in one or more dimensions. We also indicate some generalizations and applications in physics.

1.1 What is wavelet analysis?

Wavelet analysis is a particular time- or space-scale representation of signals which has found a wide range of applications in physics, signal processing and applied mathematics in the last few years. In order to get a feeling for it and to understand its success, let us consider first the case of one-dimensional signals.

It is a fact that most real life signals are nonstationary and usually cover a wide range of frequencies. They often contain transient components, whose apparition and disparition are physically very significant. In addition, there is frequently a direct correlation between the characteristic frequency of a given segment of the signal and the time duration of that segment. Low frequency pieces tend to last a long interval, whereas high frequencies occur in general for a short moment only. Human speech signals are typical in this respect: vowels have a relatively low mean frequency and last quite long, whereas consonants contain a wide spectrum, up to very high frequencies, especially in the attack, but they are very short.

Clearly standard Fourier analysis is inadequate for treating such signals, since it loses all information about the time localization of a given frequency component. In addition, it is very uneconomical: when the signal is almost flat, i.e. uninteresting, one still has to sum an infinite alternating series for reproducing it. Worse yet, Fourier analysis is highly unstable with respect to perturbation, because of its global character. For instance, if one adds an extra term, with a very small amplitude, to a linear superposition of sine waves, the signal will barely be modified, but the Fourier spectrum will be completely perturbed. This does not happen if the signal is represented in terms of *localized* components.

For all these reasons, signal analysts turn to *time-frequency* (TF) representations. The idea is that one needs *two* parameters: one, called a, characterizes the frequency, the other one, b, indicates the position in the signal. This concept of a TF representation is in fact quite old and familiar. The most obvious example is simply a musical score!

If one requires in addition the transform to be *linear*, a general TF transform will take the form:

$$s(x) \mapsto S(a, b) = \int_{-\infty}^{\infty} \overline{\psi_{ab}(x)}\, s(x)\, dx, \tag{1.1}$$

where s is the signal and ψ_{ab} the analysing function. Within this class, two TF transforms stand out as particularly simple and efficient: the Windowed or Short Time Fourier Transform (WFT) and the Wavelet Transform (WT). For both of them, the analysing function ψ_{ab} is obtained by acting on a basic (or mother) function ψ, in particular b is simply a time translation. The essential difference between the two is in the way the frequency parameter a is introduced.

(1) *Windowed Fourier Transform:*

$$\psi_{ab}(x) = e^{ix/a}\, \psi(x - b). \tag{1.2}$$

Here ψ is a window function and the a-dependence is a modulation ($1/a \sim$ frequency); the window has constant width, but the lower a, the larger the number of oscillations in the window (see Figure 1.1 (left))

(2) *Wavelet transform:*

$$\psi_{ab}(x) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{x - b}{a}\right). \tag{1.3}$$

The action of a on the function ψ (which must be oscillating, see below) is a dilation ($a > 1$) or a contraction ($a < 1$): the shape of the function is unchanged, it is simply spread out or squeezed (see Figure 1.1 (right)).

The WFT transform was originally introduced by Gabor (actually in a discretized version), with the window function ψ taken as a Gaussian; for this reason, it is sometimes called the Gabor transform. With this choice, the function ψ_{ab} is simply a canonical (harmonic oscillator) coherent state [17], as one sees immediately by writing $1/a = p$. Of course this book is concerned