Systems of Conservation Laws 1

Hyperbolicity, Entropies, Shock Waves

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Translated by

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Some models

1.1 Gas dynamics in eulerian variables

Let us consider a homogeneous gas (all the molecules are identical with mass m) in a region Ω, whose coordinates $x = (x_1, \ldots, x_d)$ are our 'independent' variables. From a macroscopic point of view, it is described by its mass density ρ, its momentum per unit volume $\vec{q}$ and its total energy per unit volume E. In a sub-domain ω containing at an instant N molecules[1] of velocities $\vec{v}^1, \ldots, \vec{v}^N$ respectively, we have

$$\int_\omega \rho \,\mathrm{d}x = Nm, \qquad \int_\omega \vec{q}\,\mathrm{d}x = m\sum_{j=1}^{N} \vec{v}^j$$

from which it follows that $\vec{q} = \rho\vec{v}$, $\vec{v}$ being the mean velocity of the molecules.[2] Likewise, the total energy is the sum of the kinetic energy and of the rotational and vibrational energies of the molecules:

$$\int_\omega E\,\mathrm{d}x = \frac{1}{2}m\sum_{j=1}^{N} \|\vec{v}^2\| + \sum_{j=1}^{N}\left(e_{\mathrm{V}}^j + e_{\mathrm{R}}^j\right)$$

where e_{V}^j and e_{R}^j are positive. For a monatomic gas, such as He, the energy of rotation is null. The energy of vibration is a quantum phenomenon, of sufficiently weak intensity to be negligible at first glance. Applying the Cauchy–Schwarz inequality, we find that

$$\frac{1}{2}m\sum_{j=1}^{N}\|\vec{v}^j\|^2 \geq \frac{m}{2N}\left\|\sum_{j=1}^{N}\vec{v}^j\right\|^2$$

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[1] N is a very large number, for example of the order of 10^{23} if the volume of ω is of the order of a unit, but the product mN is of the order of this volume.

[2] This can be suitably modified if there are several kinds of molecules of different masses.

which gives

$$\int_\omega E\,\mathrm{d}x \geq \frac{1}{2}\Big(\int_\omega \rho\,\mathrm{d}x\Big)^{-1}\Big\|\int_\omega \vec{q}\,\mathrm{d}x\Big\|^2 + \sum_{j=1}^{N}\big(e_{\mathrm{v}}^j + e_{\mathrm{R}}^j\big)$$
$$\geq \frac{1}{2}\Big(\int_\omega \rho\,\mathrm{d}x\Big)^{-1}\Big\|\int_\omega \rho\vec{v}\,\mathrm{d}x\Big\|^2.$$

This being true for every sub-domain, we can deduce that the quantity $E/\rho - \frac{1}{2}\|\vec{v}\|^2$ is positive. It is called the specific *internal energy* (that is per unit mass) and we denote it by e; we thus have

$$E = \frac{1}{2}\rho\|\vec{v}\|^2 + \rho e,$$

where the first term is (quite improperly) called the *kinetic energy* of the fluid. For the sequel it should be remembered that the internal energy can be decomposed into two terms $e_{\mathrm{k}} + e_{\mathrm{f}}$ where e_{k} is kinetic in origin and e_{f} is due to other degrees of freedom of the molecules.

The law of a perfect gas

A *perfect gas* obeys three hypotheses:

the vibration energy is null,

the velocities at a point (x, t) satisfy a gaussian distribution law

$$a\exp(-b\|\cdot - \vec{v}\|^2)$$

where a, b and $\vec{v}$ are functions of (x, t) (of course, $\vec{v}$ is the mean velocity introduced above),

the specific internal energy is made up among its different components *pro rata* with the degrees of freedom.

Comments (1) The gaussian distribution comes from the theorem of Laplace that considers the molecular velocities as identically distributed random variables when N tends to infinity. It is also the equilibrium distribution (when it is called ‘maxwellian’) in the Boltzmann equation, when it takes into account the perfectly elastic binary collisions.

(2) Several reasons characterise the gaussian as being the appropriate law. On the one hand, its set is stable by composition with a similitude O of $\mathbb{R}^d (\chi \mapsto \chi \circ O)$ and by multiplication by a scalar $(\chi \mapsto \lambda\chi)$. On the other, the components of the velocity are independent identically distributed random variables.

(3) The hypothesis of the equi-partition of energy is pretty well verified when there are a few degrees of freedom, for example for monatomic molecules (He), diatomic molecules (H_2, O_2, N_2) or rigid molecules (H_2O, CO_2, C_2H_2, C_2H_4). The more complex molecules are less rigid; they thus have more degrees of freedom, which are not equivalent from the energetic point of view.

(4) The equi-partition takes place also among the translational degrees of freedom. If the choice is made of an orthonormal frame of reference, each component $v_\alpha^j - v_\alpha$ of the relative velocity is responsible for the same fraction $e_{\mathrm{k}\alpha} = e_{\mathrm{k}}/d$ in the energy of kinetic origin.

Let β be the number of non-translational degrees of freedom. The hypothesis of equi-partition gives the following formula for each type of internal energy:

$$e_{\mathrm{k}1} = \cdots = e_{\mathrm{k}d} = \frac{1}{d} e_{\mathrm{k}}, \qquad e_{\mathrm{R}} = \frac{\beta}{d} e_{\mathrm{k}}$$

and thus $e = (d + \beta) e_{\mathrm{k}1}$.

The pressure p is the force exerted per unit area on a surface, by the gas situated on one side of it.[3] Take as surface the hyperplane $x_1 = 0$, the fluid being at rest ($\vec{v} \equiv 0$, a and b constants). Let A be a domain of unit area of this hyperplane. The force exerted on A by the gas situated to the left is proportional to the number M of particles hitting A per unit time, multiplied by the first component I_1 of the mean impulse of these.[4] On the one hand, M is proportional to the number N of particles multiplied by the mean absolute speed (the mean of $|v_1^\alpha|$) in the direction x_1. On the other hand, NI_1 is proportional to ρw_1^2, that is to $\rho e_{1\mathrm{k}}$. Nothing in this argument involves explicitly the dimension d and we therefore have $p = k\rho e_{1\mathrm{k}}$, where k is an absolute constant. A direct calculation in the one-dimensional case yields the result $k = 2$. Introducing the *adiabatic exponent*

$$\gamma = \frac{d + b + 2}{d + b}$$

there results the *law of perfect gases*

$$p = (\gamma - 1)\rho e.$$

The most current adiabatic exponents are 5/3 and 7/5 if $d = 3$, 2 and 5/3 if $d = 2$ and 3 if $d = 1$. In applications air is considered to be a perfect gas for which $\gamma = 7/5$.

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[3] In this argument, the surface in question is not a boundary, since it would introduce a reflexion and would eventually distort the gaussian distribution.

[4] This mean is not null as it is calculated solely from the set of molecules for which $v_1^j > 0$.

The Euler equations

The conservation laws of mass, of momentum and of energy can be written

$$\begin{gathered}\partial_t\rho+\operatorname{div}_x(\rho\vec{v})=0,\\ \partial_t(\rho v_i)+\operatorname{div}_x(\rho v_i\vec{v})+\partial_i p=\sum_{j=1}^{d}\partial_j T_{ij},\quad 1\le i\le d,\\ \partial_t E+\operatorname{div}_x((E+p)\vec{v})=\sum_{j=1}^{d}\partial_j(v_iT_{ij})-\operatorname{div}_x\vec{q}\end{gathered}$$

where $T-pI_d$ is the stress tensor and $\vec{q}$ the heat flux. In the last equation, two terms represent the power of the forces of stress. The conservation of the kinetic moment $\rho\vec{v}\wedge x$ implies that T is symmetric. We have seen that T is null for a fluid at rest and also when it is in uniform motion of translation. The simplest case is that in which T is a linear expression of the first derivatives $\nabla_x\vec{v}$, the coefficients being possibly functions of (ρ, e). The principle of frame indifference implies the existence of two functions α and β such that

$$T_{ij}=\alpha(\rho,e)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)+\beta(\rho,e)(\operatorname{div}_x\vec{v})\delta_i^j \tag{1.1}$$

which clearly introduces second derivatives into the above equations. The tensor T represents the effects of viscosity and the linear correspondence is Newton's law. If α and β are null the conservation laws are called *Euler's equations*. In the contrary case they are called the *Navier–Stokes equations*.

Likewise, the heat flux is null if the temperature θ (defined later as a thermodynamic potential) is constant. The simplest law is that of Fourier, which can be written

$$\vec{q}=-k(\rho,e)\nabla_x\theta,$$

with $k\ge 0$.

For a regular flow, a linear combination of the equations yields the reduced system

$$\begin{gathered}\partial_t\rho+\operatorname{div}(\rho\vec{v})=0,\\ \partial_t v_i+\vec{v}\cdot\nabla_x v_i+\rho^{-1}\partial_i p=\rho^{-1}\operatorname{div}(T_i.),\\ \partial_t e+\vec{v}\cdot\nabla_x e+\rho^{-1}p\operatorname{div}\vec{v}=\rho^{-1}\left(\sum_{i,j}T_{ij}\partial_j v_i-\operatorname{div}\vec{q}\right).\end{gathered}$$

Let us linearise this system in a constant solution, in a reference frame in which the velocity is null:

$$\begin{gathered}\partial_t R+\rho\operatorname{div}\vec{V}=0,\\ \partial_t V_i+\rho^{-1}(p_\rho\partial_i R+p_e\partial_i\chi)=\rho^{-1}(\alpha\Delta V_i+(\alpha+\beta)\partial_i\operatorname{div}\vec{V}),\\ \partial_t\chi+\rho^{-1}p\operatorname{div}\vec{V}=\rho^{-1}k\Delta(\theta_\rho R+\theta_e\chi).\end{gathered}$$

The last equation can be transformed to

$$\partial_t(\theta_\rho R + \theta_e \chi) + \lambda \operatorname{div} \vec{V} = \frac{k\theta_e}{\rho} \Delta(\theta_\rho R + \theta_e \chi).$$

A necessary condition for the Cauchy problem for this linear system to be well-posed is the (weak) ellipticity of the operator

$$(R, \vec{V}, \xi) \mapsto (0, \alpha\Delta\vec{V} + (\alpha+\beta)\nabla \operatorname{div} \vec{V},\ k\theta_e \Delta\xi)$$

which results in the inequalities

$$k\theta_e \geq 0, \qquad \alpha \geq 0, \qquad 2\alpha + \beta \geq 0. \tag{1.2}$$

The entropy

In the absence of second order terms, the flow satisfies

$$p(\partial_t \rho + \vec{v} \cdot \nabla\rho) = \rho^2(\partial_t e + \vec{v} \cdot \nabla e)$$

which suggests the introduction of a function $S(\rho, e)$, without critical point, such that

$$p\frac{\partial S}{\partial e} + \rho^2 \frac{\partial S}{\partial \rho} = 0.$$

Such a function is defined up to composition on the left by a numerical function: if $h\colon \mathbb{R} \to \mathbb{R}$ and if S works, then $h \circ S$ does too, provided that h' does not vanish. Such a function satisfies the equation

$$(\partial_t + \vec{v} \cdot \nabla)S = 0,$$

as long as the flow is regular, this signifies that S is constant along the trajectories[5] of the particles. On taking account of the viscosity and of the thermal conductivity, it becomes

$$\rho(\partial_t + \vec{v} \cdot \nabla)S = S_e \sum_{i,j} (T_{ij} \partial_j v_i) + \operatorname{div}(k\nabla\theta),$$

that is to say

$$\partial_t(\rho S) + \operatorname{div}(\rho S \vec{v}) = S_e \left(\frac{1}{2}\alpha \sum_{i,j} (\partial_i v_j + \partial_j v_i)^2 + \beta(\operatorname{div} \vec{v})^2 \right) + S_e \operatorname{div}(k\nabla\theta).$$

Free to change S to $-S$, we can suppose that S_e is strictly positive. The name *specific entropy* is given to S. The effect of the viscosity is to increase the integral

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[5] We refer to the mean trajectory.

of ρS. The second law of thermodynamics states that the thermal diffusion behaves in the same sense, that is that

$$\int_\omega S_e \operatorname{div}(k\nabla\theta)\,\mathrm{d}x \geq 0$$

if there is no exchange of heat across $\partial\omega$ (Neumann condition $\partial\theta/\partial\nu = 0$). Otherwise, this integral is compensated by these exchanges. In other terms, after integration by parts, we must have

$$\int_\omega k\nabla\theta \cdot \nabla S_e\,\mathrm{d}x \leq 0,$$

without restriction on ω. Thus $\nabla\theta \cdot \nabla S_e$ must be negative at every point and naturally for every configuration. It is then deducted that θ is a decreasing function of S_e. Free to compose θ on the left with an increasing function,[6] there is no loss of generality if we assume that $\theta = 1/S_e$, which gives the thermodynamic relation

$$\theta\,\mathrm{d}S = \mathrm{d}e + p\mathrm{d}\left(\frac{1}{\rho}\right), \quad \theta \geq 0,$$

in which $1/\theta$ appears as an integrating factor of the differential form $\mathrm{d}e + p\mathrm{d}(1/\rho)$. For a perfect gas are chosen as usual $\theta = e$ and $S = \log e - (\gamma - 1)\log\rho$.

Barotropic models

A model is *barotropic* if the pressure is, because of an approximation, a function of the density only. There are three possible reasons: the flow is isentropic or it is isothermal, or again it is the shallow water approximation.

For a regular flow without either viscosity or conduction of heat (that makes up many of the less realistic hypotheses), we have $(\partial_t + \vec{v} \cdot \nabla)S = 0$: S is constant along the trajectories. If, in addition, it is constant at the initial instant, we have $S = \text{const}$. As $S_e > 0$, we can invert the function $S(\cdot\,, \rho)$: we have $e = \mathsf{E}(S, \rho)$, with the result that also p is a function of (S, ρ). In the present context, p must be a function of ρ alone and similarly this is true of all the coefficients of the system, for example α and β. The conservation of mass and that of momentum thus form a closed system of partial differential equations (here again we have taken account of the newtonian viscosity[7]):

$$\partial_t\rho + \operatorname{div}(\rho\vec{v}) = 0,$$
$$\partial_t(\rho v_i) + \operatorname{div}(\rho v_i\vec{v}) + \partial_i p(\rho) = \operatorname{div}(\alpha(\nabla v_i + \partial_i\vec{v})) + \partial_i(\beta \operatorname{div}\vec{v})\cdot$$

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[6] This does not affect Fourier's law, as k is changed with the result that the product $k\nabla\theta$ is not.

[7] One more odd choice!

The equation of the conservation of energy becomes a redundant equation.[8] We shall use it as the ‘entropy’ conservation law of the inviscid model. We call this the *isentropic model*:

$$\partial_t \rho + \operatorname{div}(\rho \vec{v}) = 0,$$
$$\partial_t(\rho v_i) + \operatorname{div}(\rho v_i \vec{v}) + \partial_i p(\rho) = 0.$$

Its mathematical entropy is the mechanical energy $\frac{1}{2}\rho(\|\vec{v}\|^2 + e(\rho))$, associated with the ‘entropy flux’ $\rho(\frac{1}{2}\|\vec{v}\|^2 + e(\rho))\vec{v} + p(\rho)\vec{v}$. For a perfect gas, the hypothesis $S = \text{const.}$, states that $e^{\gamma-1} = c\rho$ and furnishes the *state law* $p = \kappa\rho^\gamma$. This, then, is called a *polytropic gas.*

The *isothermal* model is reasonable when the coefficient of thermal diffusion is large relative to the scales of the time and space variables. For favourable boundary conditions, the entropic balance gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho S \,\mathrm{d}x \geq -\int_\Omega k\nabla\theta \cdot \nabla S_e \,\mathrm{d}x = \int_\Omega k\frac{\|\nabla\theta\|^2}{\theta^2}\,\mathrm{d}x.$$

According to the conservation laws, we can add to ρS an affine function of the variables $(\rho, \rho\vec{v}, E)$ in the preceding inequality. Meanwhile, experience shows that the mapping $(\rho, \rho\vec{v}, E) \mapsto \rho S$ is concave.[9] We can thus choose an affine function η_0 with the result that $\eta := \rho S + \eta_0$ is negative. If the domain Ω is the whole space $\mathbb{R}^d$, the fluid being at rest at infinity, we can also take η to be null at infinity. Finally

$$\int_\Omega k\frac{\|\nabla\theta\|^2}{\theta^2}\,\mathrm{d}x \leq -\int_\Omega \eta_{t=0}\,\mathrm{d}x\cdot$$

The right-hand side is a datum of the problem, supposed finite. If k is large, we see that it is all right to approach θ by a constant; that it is a constant and not a function of time is not clear but is currently assumed. Again, the pressure and the viscosity become functions of ρ only, and the conservation of mass and that of momentum form a closed system: the mechanical energy is taken as the mathematical entropy of the system. For a perfect gas, $e = \theta$ is constant, with the result that the state law is linear: $p = \kappa\rho$.

The isothermal approximation is reasonable enough in certain régimes, because, for a gas, for instance, the thermal effects are always more significant than the viscous effects. A general criterion regarding these approximations is however that the shocks of the barotropic models are not the same as those of the Euler equations: the Rankine–Hugoniot condition is different.

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[8] Or rather incompatible, if we have included the newtonian viscosity.

[9] In fact, this concavity is the condition for the Cauchy problem of the linearised Euler equations to be well-posed. It no longer holds if we model a fluid with several phases.

The third barotropic model describes the flow in a shallow basin, that is, in one whose horizontal dimensions are great with respect to its depth. The domain Ω is the horizontal projection of the basin: we thus have $d = 1$ or $d = 2$. The fluid is *incompressible* with density ρ_0. We do not take the vertical displacements into account. The variables treated are the horizontal velocity (averaged over the height) $\vec{v}(x, t)$ and the height of the fluid $h(x, t)$. The pressure is considered to be the integral of the hydrostatic pressure $\rho_0 g z$ where z is the vertical coordinate. We therefore have $p = \rho_o g h^2/2$. The conservation of mass and that of momentum give the system

$$\partial_t(\rho_0 h) + \operatorname{div}(\rho_0 h \vec{v}) = 0,$$
$$\partial_t(\rho_0 h v_i) + \operatorname{div}(\rho_0 v_i \vec{v}) + \frac{1}{2} g \partial_i(\rho_0 h^2) = 0, \quad 1 \le i \le d.$$

Comments Dividing by ρ_0, we recover the isentropic model of a perfect gas for which $\gamma = 2$.

We have not taken into account the effects of viscosity and this is an error: they are responsible for a boundary layer on the base of the basin which implies a resistance to the motion. That resistance makes itself manifest in the model by a source term in the second equation of the form $-f(h, |\vec{v}|)v_i$, with $f > 0$.

One way of obtaining these equations from the Euler equations is to integrate the latter with respect to z (but not x). We then make the hypothesis that certain means of products are the products of means, that is that the vertical variations in ρ and $\vec{v}$ are weak.

The relativistic models of a gas, though much more complicated than those which have preceded, are also those of systems of conservation laws. We shall not give a detailed presentation here. By way of an example, we shall consider the simplest among those systems: a barotropic fluid, isentropic, one-dimensional and in special relativity; the conversation of mass and that of momentum give

$$\partial_t\left(\frac{p + \rho c^2}{c^2}\frac{v^2}{c^2 - v^2} + \rho\right) + \partial_x\left((p + \rho c^2)\frac{v}{c^2 - v^2}\right) = 0,$$
$$\partial_t\left((p + \rho c^2)\frac{v}{c^2 - v^2}\right) + \partial_x\left((p + \rho c^2)\frac{v^2}{c^2 - v^2} + p\right) = 0.$$

For more general models the reader should consult Taub [102].

1.2 Gas dynamics in lagrangian variables

Writing the equations of gas dynamics in lagrangian coordinates is very complicated if $d \ge 2$; in addition it furnishes a system which does not come into the spirit of this book. This is why we limit ourselves to the one-dimensional case ($d = 1$). We shall make a change of variables $(x, t) \mapsto (y, t)$ which depends on the solution. The conservation law of mass

$$\rho_t + (\rho v)_x = 0$$

is the only one which makes no appeal to any approximation. It expresses that the differential form $\alpha := \rho\, dx - \rho v\, dt$ is closed and therefore exact.[10] We thus introduce a function $(x, t) \mapsto y$, defined to within a constant by $\alpha = dy$. We have $dx = v\, dt + \tau\, dy$, where we have denoted by $\tau = \rho^{-1}$ the *specific volume* (which is rather a specific length here).

Being given another conservation law $\partial_t u_i + \partial_x q_i = 0$, which can be written $d\,(q_i\, dt - u_i\, dx) = 0$, we have that

$$d((q_i - u_i v)\, dt - u_i \tau\, dy) = 0,$$

that is

$$\partial_t(u_i \tau) + \partial_y(q_i - u_i v) = 0.$$

The system, written in the variables (y, t), is thus formed of conservation laws. Let us look at for example the momentum $u_2 = \rho v$. In the absence of viscosity, we have $q_2 = \rho v^2 + p(\rho, e)$. From this comes

$$\partial_t v + \partial_y P(\tau, e) = 0,$$

where $P(\tau, e) := p(\tau^{-1}, e)$. Similarly, for the energy, $u_3 = \frac{1}{2}\rho v^2 + \rho e$ and $q_3 = (u_3 + p)v$:

$$\partial_t \left(\frac{1}{2} v^2 + e\right) + \partial_y(P(\tau, e)v) = 0.$$

The conservation of mass gives nothing new since it was already used to construct the change of variables. With $u_1 = \rho$ and $q_1 = \rho v$, we only obtain the trivial equation $1_t + 0_y = 0$. To complete the system of equations for the unknowns (τ, v, e) we have to involve a trivial conservation law. For example with $u_4 \equiv 1$ and $q_4 \equiv 0$, we obtain

$$\partial_t \tau = \partial_y v.$$

We note that in lagrangian variables the perfect gas law is written $P = (\gamma - 1)e/\tau$.

If we take into account the thermal and viscous effects, then $q_2 = \rho v^2 + p(\rho, e) - \nu(\rho, e)v_x$. As $\tau v_x = v_y$ we obtain

$$\partial_t v + \partial_y P(\tau, e) = \partial_y \left(\frac{\nu}{\tau} \partial_y v\right).$$

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[10] These assertions are correct even $(\rho, \rho v)$ are no better than locally integrable.

Similarly, $q_3 = (u_3 + p)v - \nu v v_x - k\theta_x$ gives

$$\partial_t\left(\frac{1}{2}v^2 + e\right) + \partial_y(Pv) = \partial_y\left(\frac{\nu}{\tau}(v\partial_y v)\right) + \partial_y\left(\frac{k}{\tau}\partial_y\theta\right).$$

Criticism of the change of variables

Although this change of variable is perfectly justified, even if (e, v) is bounded without more regularity as well as v^{-1} (see D. Wagner [110]), it raises a major difficulty if the vacuum is somewhere part of the space. In this case, the jacobian ρ of $(x, t) \mapsto (y, t)$ vanishes and it is no longer a change of variable. The specific volume then reduces to a Dirac mass, with norm equal to the length of the interval of the vacuum. It becomes critical to give sense to the equations (it is nothing other than the conservation law of a mathematical difficulty). The equations in eulerian coordinates are also ill-posed in the vacuum: the velocity cannot be defined and the fluxes q_2 and q_3 are singular. Indeed, returning to the variables $u = (\rho, \rho v, E)$, we have $q_2 = u_2^2/u_1 + p$, which makes no sense for $\rho = 0$.

1.3 The equation of road traffic

Let us consider a highway (a unique sense of circulation will be sufficient for our purpose), in which we take no account of entries or exits. We represent the vehicle traffic as the motion of a one-dimensional continuous medium, which is reasonable if the physical domain which we consider is very great in length in comparison with the length of the cars. In normal conditions, we have a conservation law of ‘mass’

$$\partial_t\rho + \partial_x q = 0,$$

where $q = \rho v$ is the flux, or flow, and v is the mean velocity. Unlike the case of a fluid there is no conservation law of momentum or of energy. The drivers choose their velocities according to the traffic conditions. It results in a relation $v = V(\rho)$ where V is the speed limit if ρ is small. The function $\rho \mapsto V$ is decreasing and vanishes for a saturation value ρ_{m}, for which neighbouring vehicles are bumper-to-bumper. The space of the states is therefore $\mathcal{U} = [0, q_{\mathrm{m}}]$.

This model is a typical example of a scalar conservation law. The state law $q(\rho) = \rho V(\rho)$ has the form indicated in Fig. 1.1. We notice that each possible value of the flow corresponds to two possible densities, of different velocities, with the exception of the maximal flow.

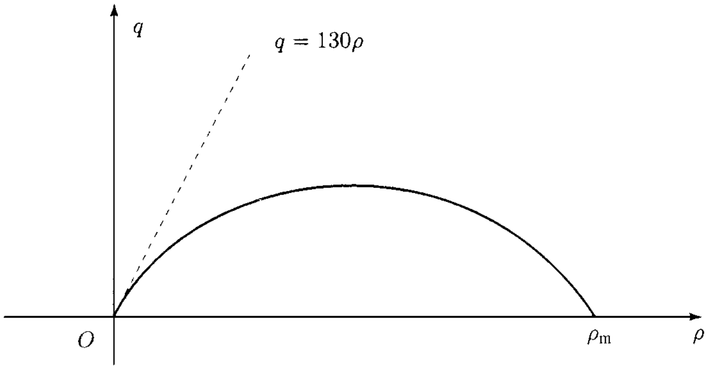

Fig. 1.1: Road traffic: flux vs density (in France).

A more precise model is obtained by taking the drivers' anticipation into account. If they observe an upstream increase in the density (respectively a diminution), they show a tendency to brake (respectively to accelerate) slightly. In other terms, $v - V(\rho)$ is of the opposite sign to that of ρ_x. The simplest state law which takes account of this phenomenon is $v = V(\rho) - \varepsilon\rho_x$, with $0 < \varepsilon \ll 1$, which leads to the weakly parabolic equation

$$\rho_t + q(\rho)_x = \varepsilon(\rho\rho_x)_x.$$

1.4 Electromagnetism

Electromagnetism is a typically three-dimensional phenomenon ($d = 3$), which brings vector fields into play: the electric intensity E, the electric induction D, the magnetic intensity H, the magnetic induction B, the electric current j and the heat flux q. Denoting by e the internal energy per unit volume, the conservation laws are

Faraday's law

$$\partial_t B + \operatorname{curl} E = 0,$$

with which is associated the compatibility condition $\operatorname{div} B = 0$ (absence of magnetic charge),

Ampère's law

$$\partial_t D - \operatorname{curl} H + j = 0,$$

conservation of energy

$$\partial_t E + \operatorname{div}(E \wedge H + q) = 0.$$

Maxwell's equations

In the first instance let us neglect the current and the heat flux (which is correct for example in the vacuum). Combining the three laws, we obtain

$$\partial_t e = H \cdot \partial_t B + E \cdot \partial_t D.$$

If the system formed by the laws of Faraday and Ampère is closed by the state laws

$$H = \mathsf{H}\,(B, D), \qquad E = \mathsf{E}\,(B, D),$$

from the conservation of energy it is then deduced that

$$\mathsf{H}\,(B, D) \cdot \mathrm{d}B + \mathsf{E}\,(B, D) \cdot \mathrm{d}D$$

is an exact differential. Following Coleman and Dill [9], we can then postulate the existence of a function $W \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ such that

$$H_j = \frac{\partial W}{\partial B_j}, \qquad E_j = \frac{\partial W}{\partial D_j}, \quad j = 1, 2, 3.$$

We have $e = W(B, D)$; the conservation laws are called *Maxwell's equations*:

$$\partial_t B + \mathrm{curl}\frac{\partial W}{\partial D} = 0, \qquad \partial_t D - \mathrm{curl}\frac{\partial W}{\partial B} = 0.$$

These lead to *Poynting's formula*

$$\partial_t W(B, D) + \mathrm{div}(E \wedge H) = 0,$$

which shows that W is an entropy of the system, generally convex. Some other entropies of the system, not convex, are the components of $B \wedge D$.

Now, taking into account the charge and the heat, the complete model is the following:

$$\partial_t B + \mathrm{curl}\,\frac{\partial W}{\partial D} = 0, \qquad \partial_t D - \mathrm{curl}\,\frac{\partial W}{\partial B} = -j,$$
$$\partial_t (W(B, D) + \varepsilon_0) + \mathrm{div}(E \wedge H + q) = 0,$$

where ε_0 is the purely calorific part of the internal energy.[11] For a regular solution we have

$$\partial_t \varepsilon_0 + \mathrm{div}\, q = E \cdot j,$$

where the right-hand side represents the work done by the electromagnetic force

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[11] We have made the hypothesis that the underlying material is fixed in the reference frame. For a material in accelerated motion, see for example the following section.

(the Joule effect). We notice that transfer between the two forms of energy is possible. In the vacuum, the current is zero and there is neither temperature, nor heat flux; next, following Feynman [21] (Chapter 12.7 of the first part of vol. II), the Maxwell equations are linear in a large range of the variables. The energy W is thus a quadratic form:

$$W(B, D) = \frac{1}{2}\left(\frac{1}{\mu_0}\|B\|^2 + \frac{1}{\varepsilon_0}\|D\|^2\right).$$

The constants of electric and magnetic permittivity have the values (in S.I. units) $\varepsilon_0 = (36\pi \cdot 10^9)^{-1}$ and $\mu_0 = 4\pi \cdot 10^{-7}$. Their product is c^{-2}, the inverse of the square of the velocity of light.

In material medium, conducting and isotropic, the state law has the same form but with constants $\varepsilon > 0$ and $\mu > 0$ of greater value. The number $(\varepsilon\mu)^{-1/2}$ is again equal to the velocity of propagation of plane waves in the medium. In media which are poor conductors (dielectrics) the state law is no longer linear. The isotropy manifests itself by the condition

$$W(RB, RD) = W(B,D), \qquad \forall R \in O_3(\mathbb{R}).$$

This implies the existence of a function w of three variables, such that

$$W(B, D) = w(\|B\|, \|D\|, B \cdot D).$$

Finally, paramagnetic bodies present phenomena of memory (with *hysteresis*), which do not come into the body of systems with conservation laws.

Plane waves

Henceforth, let us neglect the thermodynamic effects as well as the electric current. For a plane wave which is propagating in the x_1-direction we have $\partial_2 = \partial_3 = 0$, with the result that $\partial_t B_1 = \partial_t D_1 = 0$. There remain four equations, in which we write $x = x_1$, the unique space variable:

$$\partial_t B_2 - \partial_x \frac{\partial W}{\partial D_3} = 0, \qquad \partial_t B_3 + \partial_x \frac{\partial W}{\partial D_2} = 0,$$

$$\partial_t D_2 + \partial_x \frac{\partial W}{\partial B_3} = 0, \qquad \partial_t D_3 - \partial_x \frac{\partial W}{\partial B_2} = 0.$$

Let us look at the simple case in which W is a function of $\rho := (\|B\|^2 + \|D\|^2)^{1/2}$ only. Introducing the functions $y := B_2 + D_3 + \mathrm{i}(B_3 - D_2)$, $z := B_2 - D_3 + \mathrm{i}(B_3 + D_2)$, we have $y_t - (\varphi(\rho)y)_x = 0$, $z_t + (\varphi(\rho)z)_x = 0$. The polar coordinates (r, s, α, β), defined by $y = r\exp\mathrm{i}\alpha$ and $z = s\exp\mathrm{i}\beta$, enable us to simplify the system into

$$\alpha_t - \varphi(\rho)\alpha_x = 0, \qquad r_t - (\varphi(\rho)r)_x = 0,$$
$$\beta_t + \varphi(\rho)\beta_x = 0, \qquad s_t + (\varphi(\rho)s)_x = 0,$$

with the connection $2\rho^2 = r^2 + s^2$.

1.5 Magneto-hydrodynamics

Magneto-hydrodynamics (abbreviated as M.H.D.) studies the motion of a fluid in the presence of an electromagnetic field. As it is a moving medium, the field acts on the acceleration of the particles, while the motion of the charges contributes to the evolution of the field. This coupling is negligible in a great number of situations but comes into action in a Tokamak, a furnace with induction, or in the interior of a star.

The fluid is described by its density, its specific internal energy, its pressure, and its velocity. If no account is taken of the diffusion processes, we write the conservation laws of mass, of momentum, of energy and Faraday's law as follows:

$$\rho_t + \operatorname{div}(\rho v) = 0,$$
$$(\rho v_i)_t + \operatorname{div}(\rho v_i v) + \frac{\partial}{\partial x_i}\left(p + \frac{1}{2}\|B\|^2\right) - \operatorname{div}(B_i \cdot B) = 0, \quad 1 \le i \le 3,$$
$$\left(\rho\left(\frac{1}{2}\|v\|^2 + \varepsilon\right) + \frac{1}{2}\|B\|^2\right)_t + \operatorname{div}\left(\rho\left(\frac{1}{2}\|v\|^2 + \varepsilon\right) + pv + E \wedge B\right) = 0,$$
$$B_t + \operatorname{curl} E = 0.$$

We see from these equations that the magnetic field exerts a force on the fluid particles and contributes to the internal energy of the system. The fact that the electric field does not is the result of an approximation, the same as we made in disregarding Ampère's law.

There are two state laws: on the one hand $p = p(\rho, e)$, which always has the form $P = (\gamma - 1)\rho e$ for a perfect gas; on the other hand, $E = B \wedge v$. This expresses a local equilibrium: the acceleration of the particles taken individually is of the form $f + (E + v \wedge B)/m$ where m is the mass of a particle of unit charge and f is the force due to the binary interactions. As $m \ll 1$ and since the velocity of the fluid remains moderate,[12] $E + v \wedge B$ is very small.

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[12] Under this hypothesis, the fluid is seen as a dielectric.

For a sharper description, we take account of the processes of diffusion: the viscosity, Fourier's law certainly, even Ohm's law:[13]

$$E = B \wedge v + \eta j + \chi(j \wedge B).$$

Finally we take Ampère's law into account, but we neglect in it the derivative $\partial_t E$ considering that E varies slowly in time:

$$j = \operatorname{curl} B.$$

Each of the phenomena which we come to take into account is studied by adding one or several of the second order terms in the laws of conservation. Whether the factors such as η, χ, k, α and β can be considered as small or not depends on the scale of the problems studied.

Plane waves in M.H.D.

Again, we consider the solutions for which $\partial_2 = \partial_3 = 0$ and $\beta := B_1$ is constant. This behaviour is established when the initial condition satisfies it. In the sequel we write

$$z := v_1, \qquad w := (v_2, v_3), \qquad b := (B_2, B_3), \qquad x := x_1.$$

In Faraday's law $B_t + \operatorname{curl} E = 0$, the component in the direction of x_1 and the compatibility condition $\operatorname{div} B = 0$ are trivial. There remain seven equations in place of eight, which is correct since B_1 is no longer an unknown:

$$\begin{aligned}
\partial_t \rho + \partial_x(\rho z) &= 0,\\
\partial_t(\rho z) + \partial_x\left(\rho z^2 + p(\rho, e) + \frac{1}{2}\|b\|^2\right) &= 0,\\
\partial_t(\rho w) + \partial_x(\rho z w - \rho b) &= 0,
\end{aligned}$$

$$\begin{aligned}
&\partial_t\left(\rho\left(\frac{1}{2}z^2 + \frac{1}{2}\|w\|^2 + e\right) + \frac{1}{2}\|b\|^2\right) + \partial_x\left(\rho z\left(\frac{1}{2}z^2 + \frac{1}{2}\|w\|^2 + e\right)\right.\\
&\qquad \left. + (p + \|b\|^2)z - \beta b \cdot w\right) = 0,\\
&\qquad \partial_t b + \partial_x(zb - \beta w) = 0.
\end{aligned}$$

The system is simpler in lagrangian coordinates (y, t), defined by $\mathrm{d}y = \rho(\mathrm{d}x - z\,\mathrm{d}t)$ – see §1.2. Denoting by $\tau = 1/\rho$ the specific volume, these equations are

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[13] Which replaces the hypothesis $E = B \wedge v$.

transformed to

$$\begin{gathered}
\tau_t = z_y, \\
z_t + \left(p(1/\tau, e) + \frac{1}{2}\|b\|^2\right)_y = 0, \\
w_t - \beta b_y = 0, \\
\left(\frac{1}{2}z^2 + \frac{1}{2}\|w\|^2 + e + \frac{1}{2}\|b\|^2\right)_t + \left((p + \frac{1}{2}\tau\|b\|^2)z - \beta b \cdot w\right)_y = 0, \\
(\tau b)_t - \beta w_y = 0.
\end{gathered}$$

A combination of these equations gives, for a regular solution, $e_t + pz_y = 0$ or again $e_t + p\tau_t = 0$, that is to say

$$S(\tau, e)_t = 0,$$

S being the thermodynamic entropy ($\theta \, \mathrm{d}S = \mathrm{d}e + p \, \mathrm{d}\tau$). The analogous calculation in eulerian variables yields the transport equation

$$(\partial_t + z\partial_x)S = 0,$$

which shows that ρS is an entropy, in the mathematical sense, of the model.

A simplified model of waves

Let us consider the system of plane waves of M.H.D. in eulerian variables to fix the ideas, with $\beta \neq 0$. It admits in general seven distinct velocities of propagation $\lambda_1 < \lambda_2 < \cdots < \lambda_7$ among which $\lambda_4 = z$, $\lambda_2 = z - \beta\rho^{-1/2}$, and $\lambda_6 = z + \beta\rho^{-1/2}$ (λ_2 and λ_6 are the speeds of the *Alfven waves*). The four remaining speeds are the roots of the quartic equation

$$((\lambda - z)^2 - c^2)((\lambda - z)^2 - \beta^2/\rho) = (\lambda - z)^2\|b\|^2/\rho,$$

$c = c(\rho, e)$ being the speed of sound in the absence of an electromagnetic field. However, when b vanishes, we have $\lambda_3 = \lambda_2$ and $\lambda_5 = \lambda_6$. This coincidence of two speeds and the non-linearity of the equations induce a resonance. For waves of small amplitude, this phenomenon can be described by an asymptotic development.

First of all, a choice of a galilean frame of reference allows the assumption that the base state u_0, constant, satisfies $w_0 = 0$ (we already have $b_0 = 0$) and $z_0\sqrt{\rho_0} = \beta_0$. We thus have $\lambda_2(u_0) = \lambda_3(u_0) = 0$: the resonance occurs along curves (in the physical plane) with small velocities. If $u - u_0$ is of the size $\varepsilon \ll 1$ this velocity is also of the order of ε, which leads to the change of the time variable

$s := \varepsilon t$, so $\partial_t = \varepsilon\partial_s$. The other hypotheses are

on the one hand $\rho = \rho_0 + \varepsilon\rho_1 + \cdots$, $z = z_0 + \varepsilon z_1 + \cdots$, $e = e_0 + \varepsilon e_1 + \cdots$,
on the other hand $w = \sqrt{\varepsilon}\cdot(w^1(s,x) + \varepsilon w^2(s,t,x) + \cdots)$, $b = \sqrt{\varepsilon}\cdot(b^1(s,x) + \varepsilon b^2(s,t,x) + \cdots)$. We note that, although $\sqrt{\varepsilon}$ is great compared with ε, these hypotheses ensure that λ_2 and λ_3 are of the order of ε.

The examination of the terms of order ε in the conservation laws shows that ρ_1, z_1, e_1 and w^1 are explicit functions of b^1. Finally, the terms of order $\varepsilon^{3/2}$ in Faraday's law, averaged with respect to the slow variable t to eliminate b^2, furnish a system which governs the evolution of $U := b^1$:

$$\partial_t U + \sigma\partial_x(\|U\|^2 U) = 0, \tag{1.3}$$

where σ is a constant which depends only on (ρ_0, e_0). In this book, we shall copiously use the system (1.3) to illustrate the various theories, but we shall also make appeal to a slightly more general one:

$$\partial_t U + \partial_x(\varphi(\|U\|)U) = 0$$

where $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$ is a given smooth function.

1.6 Hyperelastic materials

We shall consider a deformable solid body, which occupies, at rest, a reference configuration which is an open set $\Omega \subset \mathbb{R}^d$. We describe its motion by a mapping $(x,t) \mapsto (y,t)$, $\Omega \to \mathbb{R}^d$, where y is the position at the instant t of the particle which was situated at rest at x in the reference configuration. We define the velocity $v\colon \Omega \to \mathbb{R}^d$ and the deformation tensor $u\colon \Omega \to M_d(\mathbb{R})$ by

$$v = \frac{\partial y}{\partial t}, \qquad u_{\alpha j} = \frac{\partial y_\alpha}{\partial x_j}.$$

In the first instance we write the compatibility conditions

$$\partial_t u_{\alpha j} = \partial_j v_\alpha, \qquad \partial_k u_{\alpha j} = \partial_j u_{\alpha k}, \quad 1 \le \alpha, j, k \le d.$$

A material is said to be *hyperelastic* if it admits an internal energy density of the form $W(u)$ and if the forces due to the deformation derive from this energy (principle of *virtual work*):

$$f = (f_1, \ldots, f_d), \qquad f_\alpha = -\frac{\delta E}{\delta y_\alpha}.$$

Here $\delta/\delta y$ denotes the variational derivative of $E[y] := \int W(\nabla y)\,dx$:

$$f_\alpha = \sum_{j=1} \partial_j \frac{\partial W}{\partial u_{\alpha j}}.$$

The fundamental law of dynamics is written

$$\partial_t v_\alpha = f_\alpha + g_\alpha,$$

where g represents the other forces, due to gravity or to an electromagnetic field (but here we do not consider any coupling). Finally, $U := (u, v)$ obeys a system of conservation laws of first order (for which $n = d(d+1)$)

$$\partial_t u_{\alpha j} = \partial_j v_\alpha, \quad 1 \leq \alpha, j \leq d,$$

$$\partial_t v_\alpha = \sum_{k=1}^{d} \partial_k \frac{\partial W}{\partial u_{\alpha k}} + g_\alpha, \quad 1 \leq \alpha \leq d.$$

These equations can be linear, when W is a quadratic polynomial, but this type of behaviour is not realistic. In fact, the energy is defined only for $u \in \mathrm{GL}_d(\mathbb{R})$ with $\det(u) > 0$ (the material does not change orientation), and must tend to infinity when the material is compressed to a single point:

$$\lim_{u \to 0_n} W(u) = +\infty.$$

The models of elasticity are thus fundamentally non-linear. Other restrictions on the form of W are due to the principle of *frame indifference*:

$$W(u) = W(Ru), \quad R \in \mathrm{SO}_d(\mathbb{R}), \tag{1.4}$$

and, if the material is isotropic,

$$W(u) = W(uR), \quad R \in \mathrm{SO}_d(\mathbb{R}). \tag{1.5}$$

From (1.4), there exists a function $w \colon S^+ \to \mathbb{R}$, on the cone S^+ of positive definite symmetric matrices such that

$$W(u) = w(u^{\mathrm{T}} u).$$

If, in addition, (1.5) holds, then the function $S \mapsto w(S)$ depends only on the eigenvalues of S.

We find an entropy of the system in writing the conservation of energy:

$$\partial_t \left(\frac{1}{2} \|v\|^2 + W(u) \right) = \sum_{\alpha, j=1}^{d} \partial_j \left(v_\alpha \frac{\partial W}{\partial u_{\alpha j}} \right).$$

The total mechanical energy $(v, u) \mapsto \frac{1}{2}\|v\|^2 + W(u)$ is not always convex.[14] However, it is in the 'directions compatible' with the constraint $\partial_k u_{\alpha j} = \partial_j u_{\alpha k}$. In other

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[14] There are obstructions due to the invariances mentioned above and to the fact that W tends to infinity at 0 and at infinity. See [8] Theorem 4.8-1 for a discussion.

words, W is convex on each straight line $\bar{u} + \mathbb{R}z$ where z is of rank one (the *Legendre–Hadamard condition*). This reduced concept of convexity is appropriate for problems with constraints. In particular, the mechanical energy furnishes an *a priori* estimate. A constitutive law currently used is that of St Venant and Kirchhoff:

$$w(S) = \frac{1}{2}\lambda(\operatorname{Tr} E)^2 + \mu \operatorname{Tr}(E^2), \quad E := \frac{1}{2}(S - I_n).$$

On the other hand, other entropies do not have this convexity property; for all $k \leq d$

$$\partial_t(v \cdot u_k) = \partial_k\left(\frac{1}{2}\|v\|^2 - W(u)\right) + \sum_{j=1}^{d} \partial_j\left(u_{\alpha k}\frac{\partial W}{\partial u_{\alpha j}}\right).$$

Strings and membranes

More generally, we can consider a material for which $x \in \Omega$ (with $\Omega \subset \mathbb{R}^d$) but with $(y, t) \in \mathbb{R}^p$ with $p \geq d$. The case $p = d$ is that described above. When $p = 3$ and $d = 2$ it is a mater of a membrane or a shell, while $p \geq 2$ and $d = 1$ corresponds to a string. For a membrane or a string, the equations are the same as in the preceding paragraph, but the Greek suffixes go from 1 to p instead of from 1 to d. There are then $n = p(d + 1)$ unknowns and just as many equations of evolution.

Let us look at the case of string: u is a vector and $W(u) = \varphi(\|u\|)$, because of frame indifference, φ being a state law. We have

$$\frac{\partial W}{\partial u_\alpha} = \frac{1}{r}\varphi'(r)u_\alpha, \qquad r := \|u\|,$$

with the result that $\mathrm{d}W$ is the product of φ' (called the *tension* of the string) by the unit tangent vector to it: $r^{-1}u$. There are four or six equations:

$$u_t = v_x, \qquad v_t = (r^{-1}\varphi'(r)u)_x + g.$$

1.7 Singular limits of dispersive equations

The systems of conservation laws which are presented here proceed from completely integrable dispersive partial differential equations. We take as an example the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x = u_{xxx}, \tag{1.6}$$

but there are others, of which the best known is the cubic non-linear Schrödinger equation.

Certain solutions of (1.6) are progressive periodic waves: they have the form $u = u(x - ct)$ with $u''' = 6uu' - cu'$, with the result that

$$\frac{1}{2}u'^2 = u^3 - \frac{1}{2}cu^2 - au - b,$$

where a and b are constants of integration. The triplet (a, b, c) defines a unique periodic solution (to within a translation) when the polynomial equation $P(X) := X^3 - \frac{1}{2}cX^2 - aX - b = 0$ has real roots: $u_1 < u_2 < u_3$. We then have $\min u(x) = u_1$ and $\max u(x) = u_2$.

What are of interest here are such periodic solutions of the KdV equation, which are, in first approximation, modulated by the slow variables $(s, y) := (\varepsilon t, \varepsilon x)$ with $0 < \varepsilon \ll 1$.

$$u^\varepsilon(x, t) = u_0(a(s, y), b(s, y), c(s, y); x - c(s, y)t) + \varepsilon u_1(s, y, x, t) + O(\varepsilon^2).$$

We require that u_1 and u_0 be smooth functions and that u_1 be almost periodic with respect to (x, t).

The choice of the parameters (a, b, c) is not the most practical from the point of view of calculations. We proceed to construct another set, with the aid of the expressions

$$i_1 := u, \quad i_2 := \frac{1}{2}u^2, \qquad i_3 := \frac{1}{2}u_x^2 + u^3.$$

These are invariants of the KdV equation in the sense that sufficiently smooth solutions[15] satisfy

$$\partial_t i_k + \partial_x j_k = 0, \quad 1 \le k \le 3,$$

with

$$j_1 = 3u^2 - u_{xx}, \qquad j_2 = 2u^3 + \frac{1}{2}u_x^2 - uu_{xx},$$
$$j_3 = \frac{1}{2}u_{xx}^2 + 6uu_x^2 - u_x u_{xxx} - 3u^2 u_{xx} + \frac{9}{2}u^4.$$

Let $(a_1, a_2, a_3) \in \mathbb{R}^3$ be a triplet such that there exists a function $w \in H^1(S^1)$, $S^1 = \mathbb{R}/\mathbb{Z}$, with

$$\int_0^1 w \, \mathrm{d}\xi = a_1, \quad \int_0^1 \frac{1}{2}w^2 \, d\xi = a_2, \quad \int_0^1 w^3 \, d\xi < a_3.$$

Then the set $X(a_1, a_2, a_3)$ of the couples $(v, Y) \in H^1(S^1) \times (0, +\infty)$ such that

$$\int_0^1 v \, \mathrm{d}\xi = a_1, \qquad \int_0^1 \frac{1}{2}v^2 \, \mathrm{d}\xi = a_2, \qquad \int_0^1 \left(v^3 + \frac{1}{2Y^2}v'^2\right) \mathrm{d}\xi = a_3$$

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[15] There is no interest in the question of smoothness here; let us say that it is does not cause trouble.