Systems of Conservation Laws 1

Systems of conservation laws arise naturally in several areas of physics and chemistry. To understand them and their consequences (shock waves, finite velocity wave propagation) properly in mathematical terms requires, however, knowledge of a broad range of topics. This book sets up the foundations of the modern theory of conservation laws describing the physical models and mathematical methods, leading to the Glimm scheme. Building on this the author then takes the reader to the current state of knowledge in the subject. In particular, he studies in detail viscous approximations, paying special attention to viscous profiles of shock waves. The maximum principle is considered from the viewpoint of numerical schemes and also in terms of viscous approximation, whose convergence is studied using the technique of compensated compactness. Small waves are studied using geometrical optics methods. Finally, the initial–boundary problem is considered in depth. Throughout, the presentation is reasonably self-contained, with large numbers of exercises and full discussion of all the ideas. This will make it ideal as a text for graduate courses in the area of partial differential equations.

Denis Serre is Professor of Mathematics at the Ecole Normale Supérieure de Lyon and was a Member of the Institut Universitaire de France (1992–7).

Systems of Conservation Laws 1

Hyperbolicity, Entropies, Shock Waves

DENIS SERRE

Translated by

I. N. SNEDDON

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To Paul and Fanny

Contents

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Introduction

The conservation laws that are the subject of this work are those of physics or mechanics, when the state of the system considered is a field, that is a vector-valued function $(x, t) \mapsto u(x, t)$ of space variables $x = (x_1, \dots, x_d)$ and of the time t. The domain Ω covered by x is an open set of $\mathbb{R}^d$, with in general $1 \le d \le 3$. The scalar components $u_1, \dots, u_n$ of u are variables dependent on x and t: if Ω is bounded and in the absence of any exchange with the exterior,[1] the mean state of the system

$$\bar{u} := \frac{1}{|\Omega|} \int_\Omega u(x, t)\,\mathrm{d}x$$

is independent of the time and the system tends to a homogeneous equilibrium $u \equiv \bar{u}$ as the time increases. The fact that we speak of the mean indicates that the set $\mathscr{U}$ of admissible values of the field u is a convex set of $\mathbb{R}^n$.

A conservation law is a partial differential equation

$$\frac{\partial u_i}{\partial t} + \mathrm{div}_x \vec{q}_i = g_i,$$

where $g_i(x, t)$ represents the density (per unit volume) of the interaction with external fields. Among these fields, we can even find some which depend on u; for example the conservation of momentum of an electrically neutral continuum can be written

$$\frac{\partial \rho v_i}{\partial t} + \mathrm{div}_x(\rho v_i \vec{v} - \vec{T}^i) = \rho G_i,$$

where ρ is the mass density (ρ is one of the components of u), $\mathscr{T} = (\vec{T}^1, \dots, \vec{T}^d)$ is the strain tensor, $\vec{v}$ is the velocity and $\vec{G}$ the gravity field. Hence, in general we shall

[1] The boundary $\partial\Omega$ is thus impermeable and insulated, for example electrically, in short, there is no interaction with a field other than u.

have $g_i(x,t) = h_i(x,t,u(x,t))$ where h_i is a known function: here $g_i = u_1 G_{i-1}$ since ρ is u_1. We call $\vec{h}$ the *sources* of the system.

An equivalent formulation of a conservation law is given by an integral condition, which expresses the physical balance for the quantity represented by u_i in an arbitrary part ω of Ω:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\omega u_i(x,t)\,\mathrm{d}x + \int_{\partial\omega} \vec{q}_i(x,t)\cdot\nu(x)\,\mathrm{d}x = \int_\omega g_i(x,t)\,\mathrm{d}x$$

where $\nu(x)$ denotes the outward unit normal at a point x on the boundary of ω. The vector field $\vec{q}_i$ is thus the *flux* of the variable u_i:

flux of mass if u_i is the mass density,
flux of energy if u_i is the energy density per unit volume,
electric current if u_i is the electric charge density, ...

The third formulation of a conservation law is also the most practical for finding the new equation when we have to effect a change of variables. We define a differential form α^i of degree d in $\Omega \times (0,T)$ by

$$\begin{aligned}\alpha^i := {} & u_i \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_d - q_{i1}\mathrm{d}t \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_d \\ & + \cdots (-)^d q_{id}\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{d-1}.\end{aligned}$$

The conservation law is then written

$$\mathrm{d}\alpha^i = g_i\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d.$$

This way of looking at the problem suggests that other conservation laws have a natural form $\mathrm{d}\alpha = \beta$ where α is a differential form of degree p, not necessarily equal to d. This is the case of Maxwell's electromagnetic equations or the Yang–Mills equations for which $p = 2$ and $d = 3$.

In this form, the conservation laws are intangible, in so far as the scales of time, length, velocity ... are compatible with a representation of the system by fields.[2] However, the description of the evolution of the state of the physical system is possible only if the system of equations

$$\partial_t u_i + \mathrm{div}_x \vec{q}_i = g_i, \quad 1 \le i \le n,$$

is closed under the *state laws*:

$$q_i := Q_i[u;\varepsilon].$$

These laws, in which ε denotes one or several dimensionless parameters,[3] describe

[2] Quantum effects are therefore excluded, but relativistic effects can, in general, be taken into account.
[3] Such as the inverse of a Reynolds number, a mean free path, a relaxation time.

in an empirical manner the behaviour of a continuum put into a given homogeneous state $u \in \mathscr{U}$. For example, a fixed mass of gas, in a prescribed volume and at an imposed temperature, exerts on the boundary a force whose density per unit surface area (the pressure) is constant and depends only on the thermodynamic parameters and on the nature of the gas; however, the complete ranges of time and of the space variables are not allowable and in certain cases, recourse must be had to a statistical description or to molecular dynamics. Care must always be taken, specially when an asymptotic analysis is being made, to ensure the validity of the model being used.

The description of the road traffic on a highway shows that the state law depends as much on human sciences as on physics: the average speed of vehicles is a function of the traffic density and reflects the average behaviour of human beings (the drivers) and depends on circumstances; again, there is nothing absolute about it as it varies according to material conditions (the reliability and security of the vehicles, the quality of circulation lanes), and the regulations in force and even the culture of a country.

The point common to all the models studied here is the fact that $f_i : u \mapsto Q_i[u, 0]$ is an ordinary local function: its value at (x, t) depends only on that of u at this point. By an abuse of notation, we may therefore write $f_i[u](x, t) = f_i(x, t, u(x, t))$ and, on this occasion, f_i denotes a function defined on $\mathscr{U}$ and with values in $\mathbb{R}$, which, in general, will be regular. In first approximation, the evolution of the system can be deduced from the knowledge of its initial state $u_0(\cdot) = (u_{01}, \ldots, u_{0n})$ given on Ω, in the solution of the system of non-linear partial differential equations

$$\partial_t u_i + \operatorname{div}_x f_i(u) = g_i(x, t, u), \quad 1 \le i \le n, \quad x \in \Omega, \quad t > 0, \tag{0.1}$$

augmented by the appropriate boundary conditions. This is the *mixed problem* which when $\Omega = \mathbb{R}^d$ we rather call the *Cauchy problem*.

The apparent simplicity of these equations contrasts with the difficulty of the problems encountered when solving the Cauchy problem or the mixed problem, as much from the theoretical point of view as from that of numerical analysis. These two ways of considering the Cauchy problem are equally interesting and difficult. However, the present work is devoted only to the theoretical aspects, in particular because the numerical part is covered by a number of very good books. Let us cite at least those of Leveque [62], Godlewski and Raviart [34], Richtmyer and Morton [86], Sod [98], and Vichnevetsky and Bowles [109].

To illustrate the mathematical difficulties, let us say that there is not a satisfactory result concerning the existence of a solution of the Cauchy problem. For given regular initial data,[4] there exists a regular solution, but only during a finite time

[4] Let us say of class H^s, with $s > 1 + d/2$ with the result that u_0 is of class $\mathscr{C}^1$.

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More information

(Theorem 3.6.1) inversely proportional to $\nabla_x u_0$. Since, beyond a certain time, discontinuities in u must develop, this theorem is not satisfactory for applications. The results which concern weak solutions (those which have a chance of being defined for all time) are limited to the scalar case ($n = 1$) or to the one-dimensional case ($d = 1$)! Again in this latter case the restrictions themselves are severe: the global existence in time (Theorem 5.2.1) is known if the total variation of u_0 is sufficiently small (if $n = 2$, only if the product $\mathrm{TV}(u_0)\|u_0\|_\infty$ is supposed small). There is there a threshold effect, for a local result does not exist where the time of existence would depend on the scale of the data. This question is discussed by Temple and Young [105], who have obtained recently a result of this type for the system of gas dynamics.[5] For bounded initial data, but of arbitrary size, the situation is worse; only 2×2 systems (i.e. with $n = 2$ and $d = 1$) and related systems (see Chapter 12) have been tackled by the method of *compensated compactness*, under restrictive hypotheses and for results of relatively poor quality. Among these, the *Temple systems* gain from a suitable theory (see Chapter 13) in large part because they are a faithful generalisation of the scalar laws of conservation.

The appearance of discontinuities in finite time has led specialists in function spaces to pay particular attention to spaces such as L^∞ or BV (functions of bounded variation). It is in one or the other space that existence theorems have been obtained in one-dimensional space. The reason for their success is that these are algebras, which permits the treatment of the rather strong non-linearity of the equations. However, the work of Brenner [4], which is concerned with linear systems, shows that these spaces cannot be adapted to the multi-dimensional case. To the contrary, the spaces would have to be of Hilbert type, at least to be constructed on L^2. We are thus in the presence of a paradox which has up to the present not been resolved: *to find a function space which is an algebra, probably constructed on L^2 and which contains enough discontinuous functions.*

The study of discontinuous solutions, called *weak solutions*, makes use of the integral form of the conservation laws, in the equivalent form below:[6]

$$\iint_{\Omega\times\mathbb{R}^+}\left(u_i\frac{\partial\varphi}{\partial t}+f_i(u)\cdot\nabla_x\varphi\right)\mathrm{d}x\,\mathrm{d}t+\int_\Omega u_{i0}\varphi(\cdot,u)\,\mathrm{d}x=\iint_{\Omega\times\mathbb{R}^+}g_i\varphi\,\mathrm{d}x\,\mathrm{d}t$$

for every test function φ, of class $\mathscr{C}^\infty$ and with compact support in $\Omega\times\mathbb{R}^+$. We show easily the equivalence with the partial differential equation $\partial_i u_t + \operatorname{div}_x f_i(u) = g_i$ everywhere u is of class $\mathscr{C}^1$. On the other hand, when u is of class $\mathscr{C}^1$ on both sides of a hypersurface $\Sigma \in \mathbb{R}^{d+1}$, with the boundary values $u^+(x,t)$ on one side and $u^-(x,t)$ on the other, the integral formulation expresses a transmission condition,

[5] Their work is based on the particular structure of the system of gas dynamics and cannot be extended to systems of general conservation laws, by reason of an estimation due to Joly, Métivier and Rauch [47].
[6] The eventual boundary conditions have not been taken into account here, so as not to overburden the formulae.

called the *Rankine–Hugoniot condition*:

$$\nu_0[u_i] + \sum_{\alpha=1}^{d} \nu_\alpha [f_{i\alpha}(u)] = 0$$

where $(x, t) \mapsto (\nu_0, \dots, \nu_d)$ is a normal vector field to Σ. This formula suggests that the role of the sources g_i in the propagation of discontinuities is negligible. This is the reason why these terms are omitted very frequently in this work.

A quick look at the systems of the form (0.1) suggests that they govern reversible phenomena, at least when $g \equiv 0$: if u is a solution so also is the function $\tilde{u}(x, t) := u(-x, -t)$. This is obvious if u is of class $\mathscr{C}^1$, it is also true for a weak solution. Nevertheless it is known that thermodynamics modelled by the equations of Euler which are the archetype of systems of conservation laws is the centre of irreversible processes. This paradox is bound to the lack of uniqueness of the solution of the Cauchy problem in the framework of weak solutions. The regular solutions are effectively reversible, but the discontinuous solutions are not. We attack there a central question of the theory: how to separate the wheat from the chaff, the solutions observed in nature (called 'physically admissible') from those that are only mathematical artefacts? There are two major types of reply to this question.

The first is descriptive and concerns only piecewise continuous solutions whose discontinuities occur along regular hypersurfaces of $\Omega \times \mathbb{R}^+$. These discontinuities are physically admissible if they obey a causality principle: the state of the system cannot contain more information than it has at the initial instant.[7] Mathematically, we consider the coupled system formed, on the one hand, partly of the conservation laws and partly of the hypersurface and, on the other hand, of the Rankine–Hugoniot condition, seen as an evolution equation for the location of the discontinuity. It is thus a free boundary problem, which can be transformed to a mixed problem in a fixed domain. We demand that this mixed problem be well-posed for increasing time. In dimension $d = 1$, an equivalent condition, at least if the amplitude of the jump in u is moderate, is the shock criterion of Lax, formed of four inequalities, described in §4.3. In a higher dimension, the characterisation of the admissible discontinuities, much more complex, is explained in Chapter 14. There are principally two kinds of 'good' discontinuities, according as the Lax inequalities are strict or two among them are equalities. Only the first type, called *shock waves*, are irreversible. The second type, reversible, bear the name *contact discontinuities*. Concerning thermodynamics, A. Majda [74, 73] has shown (see Chapter 14) that the shock waves are, in general, stable (that is, that the mixed problem introduced above is locally well-posed), while the contact discontinuities (the *vortex sheets*)

[7] or that the boundary conditions provide.

are strongly unstable.[8] This instability *à la Hadamard* is a stone in the garden of the mechanics of fluids; it renders Euler's equations unsuited to the prediction of flows and casts doubt on this model for thermodynamics.[9]

The second reply is of more general significance but manifests itself less practically in applications. To begin with is the criticism of the approximation made above. To simplify the matter, let us suppose ε to be a scalar. It is reasonable to replace $Q_i[u;\varepsilon]$ by $f_i(u)$ where the solution varies moderately, but it is debatable where it varies greatly. Now most often, the solution of the real problem (denoted by u^ε) is regular, of class $\mathscr{C}^1$, but varies rapidly in the neighbourhood of a discontinuity of u. Typically this neighbourhood has a width of the order of ε and the gradient of u^ε must be of the order of $1/\varepsilon$. In this narrow zone, what has been neglected is of the same order of magnitude as $f_i(u)$. We have in fact

$$Q_i[v;\varepsilon] - f_i(v) \sim \varepsilon B_i(v)\nabla_x v,$$

where $B(v)$ is a tensor with four indices. A description of the evolution more faithful than (0.1) is therefore

$$\partial_t u_i^\varepsilon + \operatorname{div}_x f_i(u^\varepsilon) = \varepsilon \operatorname{div}_x(B_i(u^\varepsilon)\nabla_x u^\varepsilon), \quad 1 \le i \le n. \tag{0.2}$$

The tensor B is such that the Cauchy problem for (0.2) is well-posed for $\varepsilon > 0$ and increasing time.[10] It represents, according to the case, the effect of a viscosity, that of thermal conduction, the Joule effect, ... In the model of road traffic, where the scalar u is the density of the vehicles, it represents the faculty of anticipation of the drivers as a function of the flow of traffic in the vicinity of their vehicle; it is this anticipation that causes irreversibility.

The system (0.2) is irreversible. This is the essential difference from (0.1), which is expressed quantitatively as follows. The undisturbed system is in general compatible, for the regular solutions, with a supplementary conservation law[11]

$$\partial_t E(u) + \operatorname{div}_x F(u) = \mathrm{d}E(u) \cdot g(x,t,u) \tag{0.3}$$

where $E\colon \mathscr{U} \to \mathbb{R}$ is strictly convex. This can always be reduced to the case in which E has positive values. The equation (0.3) then yields an *a priori* estimate of

[8] save in dimension $d = 1$ where the contact discontinuities are stable.

[9] It is not difficult to see that the vortex sheets necessarily appear, if $d > 2$, as a by-product of the interaction of multi-dimensional shocks. The case $d = 1$ is less clear.

[10] It would be wrong nevertheless to believe that the system (0.2) is parabolic, that is, that the operator $v \mapsto \operatorname{div}_x B(v)\nabla_x v$ is elliptic. Its symbol is generally positive but not positive definite.

[11] It is principally in this setting that this work is placed.

u in a Sobolev–Orlicz space via the differential equation[12]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(u(x,t))\,\mathrm{d}x = \int_\Omega \mathrm{d}E(u)\cdot g(x,t,u)\,\mathrm{d}x.$$

This allows us to control the value of the positive expression

$$\mathscr{E}(t) := \int_\Omega E(u(x,t))\,\mathrm{d}x.$$

For example, in the absence of sources, $\mathscr{E}(t)$ remains equal to $\mathscr{E}(0)$, which depends only on the initial condition. Of course, this calculation, in which we differentiate composite functions (for example $E \circ u$), does not have a rigorous basis for weak solutions. On the other hand, the solution of the perturbed problem is in general regular and satisfies the equation

$$\partial_t E(u^\varepsilon) + \mathrm{div}_x \vec{F}(u^\varepsilon) = \mathrm{d}E(u^\varepsilon)\cdot(\varepsilon\,\mathrm{div}_x(B(u^\varepsilon)\nabla_x u^\varepsilon) + g(x,t,u^\varepsilon)).$$

If $g \equiv 0$ and if the boundary values are favourable, we obtain at the best

$$\mathscr{E}'(t) = -\varepsilon\int_\Omega (\mathrm{D}^2 E(u^\varepsilon)\nabla_x u^\varepsilon \cdot B(u^\varepsilon)\nabla_x u^\varepsilon)\,\mathrm{d}x$$

which is negative for all the realistic examples. But, above all, the right-hand side does not tend to zero with ε, because we integrate an expression of the order of ε^{-1} (because of $\varepsilon(\nabla_x u^\varepsilon)^2$) over a zone whose measure is of the order of ε. The decay of $\mathscr{E}$, certainly preserved by passage to the limit when $\varepsilon \to 0^+$, thus will be strict in the presence of discontinuities. A criterion of the admissibility of solutions is thus

$$\iint_{\Omega\times\mathbb{R}^+} (E(u)\partial_t\varphi + F(u)\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} E(u_0)\varphi(\cdot,0)\,\mathrm{d}x \geq 0 \qquad (0.4)$$

for every *positive* test function $\varphi \in \mathscr{D}(\Omega\times\mathbb{R})$, with equality for a classical solution of (0.1). On the level of the discontinuities, (0.4) is translated as the jump condition[13]

$$\nu_0[E(u)] + \sum_{\alpha=1}^{d} \nu_\alpha[F_\alpha(u)] \leq 0. \qquad (0.5)$$

The equality in (0.5) is in general incompatible with the Rankine–Hugoniot condition except where this concerns the contact discontinuities.

[12] To simplify the exposition, no account has been taken of the boundary conditions. For example, the reader could assume that $\vec{F}\cdot\nu$ is null on the boundary.
[13] We remark that this condition is independent of the orientation of Σ.

A function E like that introduced above carries the name, in so far as it is a mathematical object, of *entropy*. By extension, we call a function E, not necessarily convex, in a conservation law compatible with (0.1), also an entropy. The vector field $\vec{F}$ is called the *entropy flux* associated with E. Again, this terminology is due to thermodynamics, as the form of the equations of motion in *lagrangian*[14] variables is a system of conservation laws compatible with a supplementary law in which E is equal to $-S$, the opposite of physical entropy of the fluid. This change of sign, which renders convex that which is concave and conversely, is a cultural difference between mathematicians and physicists. For the physicist, the entropy has a tendency to increase, while for the mathematician the opposite holds. In the eulerian representation, in which Ω is a domain in physical space, the difference is still more marked, as E corresponds to $-\rho S$. Despite this, the historical link between the physical theory and mathematics has led to the inequality (0.4) being called the *entropy condition*, when a system of conservation laws can be modelled on something other than the flow of a fluid. The weak solutions which satisfy (0.4) are called *entropy solutions*. By extension and in an improper manner, we again speak of entropy conditions with regard to the Lax shock condition, principally because in thermodynamics, the Lax inequalities express the fact that the entropy S, constant when it follows a particle, in fact grows when it crosses a shock wave.

The mechanism of dissipation, which makes $\mathscr{E}$ decrease and renders the evolution irreversible, is so central that it would not make sense to study theoretically (0.1), in isolation. This necessarily leads to the algebraic notion of a hyperbolic system in the linear case, but the understanding of the non-linear case calls for as much attention to be paid to the (partially) parabolic (0.2). This is why this book is not entitled *Hyperbolic systems of conservation laws*. Chapters 6, 7 and 15 are principally devoted to parabolic systems and these are involved in a significant way in Chapters 8 and 9.

Some references This volume owes a great deal to those which preceded it, in particular that of Majda [75]; this, at the same time short and profound, remains an essential reference and the energy which animates it gives birth to a sense of vocation. It is the only one to deal with nearly all the topics which deal with multi-dimensional or asymptotic problems. It is with this that we have tried to deal here, with more detail but less animation. Although dealing with many subjects, this work does not go on as long on classical problems as more specialised works. Thus, the reader who wishes to deepen his knowledge of the Riemann problem should read

[14] In these variables, a particle is represented by a fixed value of the variable x. This is therefore not a 'space variable' as strictly defined.

the text of Ling Hsiao and Tong Zhang [46]. The global methods, based on the Conley index, for studying the viscosity profiles, are found in Smoller [97]. A systematic study of the propagation and the interaction of non-linear waves is greatly developed by Whitham [112]; see also the monograph of Boillat [2]. For questions concerning the mechanics of fluids, with the description of multi-dimensional shocks, Courant and Friedrichs [11] should be consulted. Various types of singular perturbations (models of combustion, the incompressible limit) and the stability of multi-dimensional shocks are presented in Majda [75]. For mixed problems in a (partly) parabolic context, a good reference is Kreiss and Lorenz [55]. In the lecture notes by Hörmander [45] is found a simple presentation of the blow-up mechanisms for a general system (not necessarily rich) as well as the global (or nearly global) existence for a perturbation of the wave equation in dimension $d \geq 2$. The notes of Evans [20] give a view of the methods utilising weak convergence, which goes beyond mere compensated compactness. The decay of entropic solutions to N-waves is the subject of the memoir of Glimm and Lax [33]. Concerning the Cauchy problem and mixed problems for linear equations there are many references; let us, at least, cite Ivrii [48], Sakamoto [87] and again Kreiss and Lorenz [55]. The quasi-linear mixed problem in dimension $d = 1$, which includes the free boundary problems, is systematically studied in Li Ta-Tsien and Yu Wen-ci [65]. The geometrical aspects of the conservation laws, especially affine and convex, are the subject of the memoir of Sévennec [93].

The way the chapters of this book are ordered is merely an indication to the reader, since the chapters depend little on one another. The core of the theory is constituted by Chapters 2, 3, 4, 6 and 14. For a postgraduate course in which the aim is the solution of the Riemann problem for gas dynamics, Chapters 2 and 4 are indispensable but are not enough to give an advanced student a representative picture of the subject.

In spite of its length, this work does not pretend to be exhaustive. It leads to a blind alley on several questions, of which some are important. Uniqueness is the most important of these; the reason is that it is a matter of a subject which is much less advanced than that of existence (with, however, recent progress by A. Bressan), and on which we could not give a synthetic view. Likewise, this book does not tackle questions which touch on ‘pathology’: systems not strictly hyperbolic have other conditions for the admission of shock waves. At times, it has been mathematical rigour that has been neglected (with the hope that it is not too frequent for the taste of the reader): above all an attempt has been made to be the most descriptive possible, giving perhaps too many criteria and formulae, asymptotic analysis, and not enough proofs. Some new results will be found (few enough and none major in all cases) and lists of exercises which should satisfy those who believe in acquiring insight by the solution of examples. In spite of this range of descriptive material, there is not a word on the phenomenon of relaxation, nor on kinetic formulations, and not more on the description by N-waves of behaviour in large times, three important subjects of the theory. Perhaps, if the occasion arises, a future edition . . .

Lyon, November 1995

. . . cela peut durer pendant très longtemps,
si l'on ne fait pas d'omelette avant!

(Robert Desnos, Chantefables)