0. DESCRIPTIVE SET THEORY

In this book we will assume that the reader is familiar with the basic notions and results of descriptive set theory, which can be found in Moschovakis [80] and Kechris [95].

In particular, a Polish space is a separable completely metrizable topological space. The Borel sets of a topological space are those generated by the open sets by the operations of complementation and countable union. A standard Borel space is a measurable space, i.e., a set X equipped with a σ-algebra $\mathcal{S}$, such that there exists a Polish topology on X with $\mathcal{S}$ its σ-algebra of Borel sets. A countably separated measurable space is a measurable space admitting a countable family of measurable sets which separate points.

We will often make use of the Effros Borel space on the class $\mathcal{F}(X)$ of all closed subsets of a Polish space $X$. This is the measurable space generated by the sets

$$\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$$

for $U \subseteq X$ open. It is a standard Borel space.

The class of Borel sets in a separable metrizable space ramifies into a transfinite hierarchy of Borel classes $\Sigma^0_\alpha, \Pi^0_\alpha, \Delta^0_\alpha, 1 \leq \alpha < \omega_1$, defined as follows:

- $\Sigma^0_1 = \text{open}$, $\Pi^0_1 = \text{closed}$,
- $\Sigma^0_\alpha = \{ \bigcup_{n \in \mathbb{N}} A_n : A_n \in \Pi^0_{\beta_n} \text{ for some } \beta_n < \alpha \}$,
- $\Pi^0_\alpha = \text{the class of complements of } \Sigma^0_\alpha \text{ sets.}$

Also let

$$\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$$

be the class of ambiguous sets at level $\alpha$. We have $\Sigma^0_\alpha \cup \Pi^0_\alpha \subseteq \Delta^0_\beta$ for $\alpha < \beta$ and $\bigcup_{\alpha < \omega_1} \Sigma^0_\alpha$ is the class of Borel sets. We call a function between separable metrizable spaces Borel-measurable if the inverse image of a Borel set is Borel. This notion clearly makes sense for standard Borel spaces as well.

We denote by $\Sigma^1_1$ the class of analytic sets in a Polish space, i.e., the continuous images of Borel sets. Let also $\Pi^1_1$ be the class of co-analytic
sets, i.e., the complements of analytic sets. We define the \textit{projective} classes $\Sigma^1_n, \Pi^1_n, \Delta^1_n$, for $1 \leq n < \omega$, by induction as follows:

\[
\begin{align*}
\Sigma^1_{n+1} &= \text{the class of continuous images of } \Pi^1_n \text{ sets}, \\
\Pi^1_{n+1} &= \text{the class of complements of } \Sigma^1_n \text{ sets}, \\
\Delta^1_n &= \Sigma^1_n \cap \Pi^1_n.
\end{align*}
\]

Thus, by Souslin’s Theorem,

\[
\Delta^1_1 = \text{Borel}.
\]

We call a separable metrizable space \textit{analytic} if it is a homeomorphic to an analytic set in a Polish space. Similarly a measurable space is \textit{analytic} if it is isomorphic as a measurable space to the Borel space of an analytic space.

Beginning in \S7, we will also make occasional use of notions of effective descriptive set theory for which we refer the reader to Moschovakis [80]. In particular, we use the “lightface” notation $\Sigma^0_\alpha, \Pi^0_\alpha, \Delta^0_\alpha$ (for $\alpha$ a recursive ordinal), $\Sigma^1_\alpha, \Pi^1_\alpha, \Delta^1_\alpha$ for the effective classes corresponding to the classical classes $\Sigma^0_\alpha, \Pi^0_\alpha, \Delta^0_\alpha, \Sigma^1_\alpha, \Pi^1_\alpha, \Delta^1_\alpha$ for which we employ the usual “boldface” notation.
1. POLISH GROUPS

We will survey here some basic facts from the theory of Polish groups.

1.1 Metrizable groups

A topological group is a group \((G, \cdot, e)\) together with a topology on \(G\) such that \((x, y) \mapsto xy^{-1}\) is continuous (from \(G^2\) into \(G\)). We first state a standard result on the metrizability of topological groups.

1.1.1 Theorem. (Birkhoff–Kakutani) Let \(G\) be a topological group. Then \(G\) is metrizable iff \(G\) is Hausdorff and the identity \(e\) of \(G\) has a countable nbhd basis. Moreover, if \(G\) is metrizable, then \(G\) admits a compatible metric \(d\) which is left-invariant, i.e.,

\[ d(gx, gy) = d(x, y). \]

(Similarly, a right-invariant such metric exists.)

For a proof, see Berberian [74, p. 28].

The next result describes the completion process for metric groups.

1.1.2 Theorem. Let \(G\) be a topological group with compatible left-invariant metric \(d\). Let

\[ D(x, y) = d(x, y) + d(x^{-1}, y^{-1}). \]

Then \(D\) is also a compatible metric on \(G\) and if \((\hat{G}, \hat{D})\) is the completion of \((G, D)\), then the multiplication operation extends uniquely to \(\hat{G}\), so that it becomes a topological group in the topology induced by \(\hat{D}\).

For a proof, see Topsøe and Hoffmann-Jørgensen [80].

1.2 Polish groups

A Polish group is a topological group whose topology is Polish.

Thus a Polish group admits a compatible left-invariant metric. However, it may not admit a compatible left-invariant complete metric.

It follows from 1.1.2 that any metrizable topological group is densely embedded in a completely metrizable one and thus any separable metrizable topological group is densely embedded in a Polish one.
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The following is a basic fact concerning subgroups of a Polish group.

1.2.1 Proposition. A subgroup $H$ of a Polish group $G$ is Polish (in the relative topology) iff $H$ is $G_δ$ iff $H$ is closed.

Proof. The equivalence of the first two conditions is a standard result in topology. Clearly every closed subgroup of $G$ is Polish. If now $H$ is $G_δ$, consider $\overline{H}$, which is a closed subgroup of $G$, thus Polish. Replacing $G$ by $\overline{H}$ we can thus assume that $H$ is dense in $G$ and show that $H = G$. Since $H$ is a dense $G_δ$ in $G$, so is every coset of $H$. By the Baire Category Theorem, $H = G$.

Remark. By the same argument, if $G$ is a Polish group and $H \subseteq G$ is a comeager subgroup, then $H = G$.

1.2.2 Corollary. Let $G$ be a Polish group with compatible left-invariant metric $d$. Then

$$D(x, y) = d(x, y) + d(x^{-1}, y^{-1})$$

is a compatible complete metric for $G$. (In particular, if $d$ is both left- and right-invariant, $d$ is complete.)

Proof. By 1.1.2, $G$ is a Polish subgroup of $\hat{G}$, so it is closed in $\hat{G}$.

For each closed subgroup $H$ of a Polish group $G$ denote by $G/H$ the set of left-cosets of $H$ and by $\pi_H : G \to G/H$ the canonical projection $\pi_H(x) = xH$. The quotient topology on $G/H$ consists of all $U \subseteq G/H$ such that $\pi_H^{-1}(U)$ is open in $G$. Then $\pi_H$ is both continuous and open. If $d$ is a right-invariant compatible metric on $G$, then

$$d^*(xH, yH) = \inf\{d(u, v) : u \in xH, v \in yH\}$$

is a compatible metric for $G/H$, so $G/H$ is separable metrizable (see Bourbaki [66, IX, §3, 1]). Since $G/H$ is the continuous open image of a Polish space, by a result of Sierpinski (see Kechris [95, 8.19]), $G/H$ is a Polish space too. In particular, if $H$ is normal, $G/H$ is a Polish group (and the metric $d^*$ defined above is also right-invariant).

We summarize:

1.2.3 Proposition. Let $G$ be a Polish group and $H \subseteq G$ a closed subgroup. Then $G/H$ (with the quotient topology) is a Polish space, and so if $H$ is also normal, $G/H$ is a Polish group.

We also have the following basic fact concerning the coset space of a closed subgroup.
1.2 Polish groups

1.2.4 Theorem. (Dixmier) Let $G$ be a Polish group and $H$ a closed subgroup. Then there is a Borel-measurable function $s_H : G/H \to G$ such that $s_H(xH) \in xH$, i.e., $s_H$ is a Borel selector for the (left-) cosets of $H$. In particular, there is a Borel transversal $T_H$ for $H$, i.e., a Borel set meeting every such coset in exactly one point. Moreover, we have the following uniformity: For each $H$ we can find $T_H$ so that if we let

$$ T(x, H) \iff H \text{ is a closed subgroup and } x \in T_H, $$

then $T \subseteq G \times \mathcal{F}(G)$ is Borel, where $\mathcal{F}(G)$ is the Effros Borel space of the closed subsets of $G$.

For a proof, see, e.g., Kechris [95, 12.17]. (The uniformity can be easily derived from that proof.)

If $H$ is a subgroup of a Polish group $G$, we can define the quotient $\sigma$-algebra $S_H$ on $G/H$ by:

$$ A \in S_H \iff \pi_H^{-1}(A) \text{ is Borel}. $$

If $H$ is closed, it is not hard to see, using 1.2.4, that $S_H$ is the $\sigma$-algebra of Borel sets of the Polish space $G/H$. In particular, $S_H$ is countably separated. Miller [77] has conversely proved that if $S_H$ is countably separated, $H$ is closed.

From either 1.2.3 or 1.2.4 it follows that if $H$ is a closed subgroup of a Polish group $G$, then the index $[G : H]$ is either $\leq \aleph_0$ or $2^{\aleph_0}$. Notice also that if $G$ is a Polish group and $H \subseteq G$ is an open subgroup, then $H$ is actually clopen and $[G : H] \leq \aleph_0$. Thus for a closed subgroup $H$ of a Polish group $G$, $[G : H] \leq \aleph_0$ iff $H$ is clopen. It follows also from 1.2.5 below that any subgroup $H$ of a Polish group $G$ which has the property of Baire and is not meager is actually clopen (and vice versa of course).

Finally, notice that if $G$ is a Polish group and $H \subseteq G$ is a meager subgroup of $H$, then $[G : H] = 2^{\aleph_0}$. (This generalizes a result in Hodges–Hodkinson–Lascar–Shelah [93].) To prove this we apply the result of Mycielski, Kuratowski that for any perfect Polish space $X$ and for any comeager $R \subseteq X^2$, there is a Cantor set $C \subseteq X$ such that if $x, y \in C$ and $x \neq y$, then $(x, y) \in R$ (see Kechris [95,19.1]): Let $H \subseteq G$ be a meager subgroup. Then $G$ is perfect (otherwise it would be discrete), and let $R = \{(g, h) : gh^{-1} \not\in H\}$. Since $(g, h) \mapsto gh^{-1}$ is continuous and open, $R$ is comeager. So there is a Cantor set $C \subseteq G$ such that for $g, h \in C$ and $g \neq h$, $gh^{-1} \not\in H$, so $g, h$ belong to different cosets of $H$ and we are done. (This argument is due to Solecki.)
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We next discuss homomorphisms. First notice that if $G, H$ are Polish groups and $S \subseteq G$ is a dense subgroup, then any continuous homomorphism $\varphi : S \to H$ extends uniquely to a continuous homomorphism $\varphi^* : G \to H$. To see this, let $d$ be a left-invariant compatible metric for $H$ and let $D(x, y) = d(x, y) + d(x^{-1}, y^{-1})$ be the corresponding complete compatible metric for $H$ by 1.2.2. It is enough to show that if $d_n \in S$, $d_n \to g \in G$, then $\varphi(d_n)$ is $D$-Cauchy in $H$ (since then we can define: $\varphi^*(g) = \lim \varphi(d_n)$). This amounts to showing that $d(\varphi(d_n), \varphi(d_m)), d(\varphi(d_n^{-1}), \varphi(d_m^{-1})) \to 0$ as $n, m \to \infty$, or equivalently that $d(\varphi(d_n^{-1}d_m), e), d(\varphi(d_md_n^{-1}), e) \to 0$, which easily follows from the fact that $d_n \to g$, $d_n^{-1} \to g^{-1}$ and $G$ admits a left-invariant compatible metric.

Next notice that if $G, H$ are Polish groups and $\varphi : G \to H$ is a continuous homomorphism, the canonical map $\varphi^* : G/\ker(\varphi) \to H$ is a continuous injection of the Polish group $G/\ker(\varphi)$ into $H$ with range $\varphi(G)$. In particular, $\varphi(G)$ is a Borel subgroup of $H$.

**Remark.** This last fact immediately gives, for example, an affirmative answer to the question raised after Theorem 1.3 in Macpherson–Woodrow [92].

We now recall the following basic fact (for a proof, see Kechris [95, 9.9])

**1.2.5 Theorem.** (Pettis) Let $G$ be a topological group and $A \subseteq G$ a set having the property of Baire which is not meager. Then $A^{-1}A$ contains an open nbhd of $e$.

From this the following is an easy consequence.

**1.2.6 Theorem.** Let $\varphi : G \to H$ be a Baire measurable homomorphism between Polish groups. Then $\varphi$ is continuous. If moreover, $\varphi(G)$ is not meager, $\varphi$ is also open.

In particular, a surjective continuous homomorphism $\varphi$ of $G$ onto $H$ is continuous and open and so the induced map $\varphi^* : G/\ker(\varphi) \to H$ is a homeomorphism, so an isomorphism of the topological groups $G/\ker(\varphi)$ and $H$. (An isomorphism between topological groups is an algebraic isomorphism which is also a homeomorphism.)

It also follows that any Borel-measurable algebraic isomorphism between Polish groups is necessarily an isomorphism. In particular, if a group $G$ admits two Polish topologies under which it is a topological group and these topologies have the same Borel sets, these topologies are identical.

Finally, it is easy to verify that if $(G_n)$ is a sequence of Polish groups, so is $\prod_n G_n$. In particular, if $G$ is a Polish group, so is $G^N$. 
1.3 Examples

There is a great variety of Polish groups that occur in many areas of mathematics. Here is a small sample of them:

(i) Every Hausdorff second countable locally compact group is Polish.

These include all countable groups (with the discrete topology), \((\mathbb{R}^n, +), (\mathbb{T}, \cdot) (\mathbb{Z}_n^N, +)\) (the Cantor group), Lie groups, like \(GL(n, \mathbb{C})\) (the group of \(n \times n\) nonsingular complex matrices), \(U(n)\) (the compact subgroup of \(GL(n, \mathbb{C})\) of unitary matrices), etc.;

(ii) \((X, +), \) where \(X\) is a separable Banach space;

(iii) \(S_\infty,\) the symmetric group on a countable infinite set, say \(\mathbb{N},\) with the topology it inherits as a \(G_\delta\) subset of the Baire space \(\mathcal{N} = \mathbb{N}^\mathbb{N};\)

(iv) The unitary group \(U(H),\) of a separable infinite-dimensional (complex) Hilbert space \(H,\) with the weak (or equivalently) strong operator topology, i.e., the infinite-dimensional analog of \(U(n).\)

(v) The group of homeomorphisms \(H(X)\) of a compact metrizable space with the topology it inherits as a \(G_\delta\) subset of \(C(X, X);\)

(vi) The group of isometries \(\text{Iso}(X, d)\) of a complete separable metric space with the topology induced by the evaluation functions \(f \mapsto f(x), x \in X;\)

(vii) If \(X\) is a Polish space and \(\mu\) a Borel probability measure on \(X,\) let \(\text{Aut}(X, \mu)\) be the group of Borel automorphisms \(T\) of \(X,\) modulo \(\mu\)-measure 0, which preserve \(\mu,\) i.e., \(T\mu = \mu.\) Similarly, let \(\text{Aut}^*(X, \mu)\) be the group of such \(T\) that preserve the measure class of \(\mu,\) i.e., \(T\mu \sim \mu.\) Thus \(\text{Aut}(X, \mu) \subseteq \text{Aut}^*(X, \mu).\) With appropriate natural topologies, see e.g., Kechris [95, 17.46], \(\text{Aut}(X, \mu), \text{Aut}^*(X, \mu)\) are Polish groups, in fact can be viewed (up to isomorphism) as closed subgroups of \(U(L^2(X, \mu));\)

(viii) If \(G\) is a Polish group and \(X\) is compact metrizable, then \(C(X, G)\) under pointwise multiplication is a Polish group.

1.4 Universal Polish groups

Among all Polish groups there is a “largest” one. This is the content of the next theorem.

1.4.1 Theorem. (Uspenskii) Every Polish group is isomorphic to a (necessarily closed) subgroup of \(H(I^N),\) the group of homeomorphisms of the Hilbert cube \(I^N, I = [0, 1].\)
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For a proof, see Uspenskii [86] or Kechris [95, 9.18].

The following however remains open.

1.4.2 Open problem. Is there a Polish group $G$ such that every Polish group is isomorphic to $G/N$ for some closed normal subgroup $N$ of $G$?

For more on this, see Kechris [94].

1.5 Some facts about the symmetric group $S_\infty$

This group and its closed subgroups play an important role in model theory.

As mentioned earlier, the group $S_\infty$ of all bijections of $\mathbb{N}$ is a $G_\delta$ subset of the Baire space $\mathcal{N}$ and a topological group under the relative topology, so is a Polish group.

If $d$ is the usual metric on $\mathcal{N}$, i.e., $d(x,y) = 2^{-n-1}$, if $x \neq y$ and $n$ is least with $x(n) \neq y(n)$, then $d$ is a left-invariant compatible metric on $S_\infty$ and $D(x,y) = d(x,y) + d(x^{-1},y^{-1})$ is a compatible complete metric on $S_\infty$. (It is not hard to see that $S_\infty$ admits no compatible left-invariant complete metric.)

If $b$ is a bijection between finite subsets of $\mathbb{N}$, let

$$N_b = \{ g \in S_\infty : g \text{ extends } b \}.$$ 

Then $\{N_b\}$ is a basis for $S_\infty$ which is both left-invariant and right-invariant, in the sense that for any $g \in S_\infty$, $gN_b$ and $N_bg$ are also in this basis. Thus $S_\infty$, and therefore all its closed subgroups, admits a countable basis which is both left-invariant and right-invariant.

If for each finite subset $S \subseteq \mathbb{N}$, we let $N_{(S)} = N_{id_S}$ be the pointwise stabilizer of $S$, i.e., the group of all permutations leaving $S$ fixed pointwise, then $\{N_{(S)}\}$ is a countable nbhd basis of the identity of $S_\infty$ which consists of open subgroups. (The basis $\{N_b\}$ is clearly the family of left-cosets corresponding to the open subgroups $\{N_{(S)}\}$.) Thus $S_\infty$, and all its closed subgroups, admits a countable nbhd basis at the identity consisting of open subgroups.

Finally, notice that the canonical metric $d$ on $S_\infty$ introduced earlier is an ultrametric, i.e., $d(x,z) \leq \max\{d(x,y),d(y,z)\}$, thus $S_\infty$, and all its closed subgroups, admits a compatible left-invariant ultrametric.

It turns out that each one of the above properties characterizes the closed subgroups of $S_\infty$. 
1.5 Some facts about the symmetric group $S_\infty$

1.5.1 Theorem. Let $G$ be a Polish group. Then the following are equivalent:

(i) $G$ is isomorphic to a closed subgroup of $S_\infty$;

(ii) $G$ admits a countable nbhd basis at the identity consisting of open subgroups;

(iii) $G$ admits a countable basis closed under left multiplication (or a countable basis closed under right multiplication);

(iv) $G$ admits a compatible left-invariant ultrametric.

Proof. (i) $\Rightarrow$ (ii) - (iv) were already noted.

(ii) $\Rightarrow$ (iii): The left-cosets of the elements of the nbhd basis form a countable basis closed under left multiplication.

(iii) $\Rightarrow$ (i): If $G$ is finite the result is clear. So assume $G$ is infinite and let $\{U_0, U_1, U_2, \ldots \}$ be a one-to-one enumeration of a basis closed under left multiplication. To each $g \in G$ assign the element $\pi_g \in S_\infty$ given by

$$\pi_g(m) = n \Leftrightarrow gU_m = U_n.$$  

By 1.2.6 $g \mapsto \pi_g$ is a continuous injective homomorphism of $G$ into $S_\infty$. To show that it is a homeomorphism, it is enough to show that if $g_n \in G$ and $\pi_{g_n} \to \pi \in S_\infty$, then $\{g_n\}$ converges in $G$. Let $d$ be a left-invariant compatible metric on $G$ and $D(x,y) = d(x,y) + d(x^{-1}, y^{-1})$ the corresponding complete compatible metric.

It is then enough to show that $d(g_n,g_m) \to 0$ and $d(g_n^{-1}, g_m^{-1}) \to 0$, as $n,m \to \infty$, or equivalently $d(g_n^{-1}g_m, e) \to 0$ and $d(g_mg_n^{-1}, e) \to 0$. Let $\varepsilon > 0$ be given. Find $U_k$ such that $U_kU_k^{-1} \subseteq \{g : d(g, e) < \varepsilon\}$. Then, since $\pi_{g_n} \to \pi \in S_\infty$, find $N$ so that for $n,m \geq N$, $\pi_{g_n}(k) = \pi_{g_m}(k) = \pi(k)$. Then $g_nU_k = g_mU_k$, so $g_m^{-1}g_n \in U_kU_k^{-1} \subseteq \{g : d(g, e) < \varepsilon\}$, thus $d(g_m^{-1}g_n, e) < \varepsilon$. So $d(g_m^{-1}g_n, e) \to 0$. The argument that $d(g_mg_n^{-1}, e) \to 0$ is similar, using the fact that $(\pi_{g_n})^{-1} = \pi_{g_n}^{-1} \to \pi^{-1}.$

(iv) $\Rightarrow$ (ii): If $d$ is such a metric, note that $\{g \in G : d(g, e) < 1/n\} = U_n$ is a nbhd basis of the identity of $G$ and consists of open subgroups.

Remark. Recall that a Polish space $X$ is zero-dimensional if it admits a compatible ultrametric (see, e.g., Kechris [95, 7.1]). Also the Baire space $\mathcal{N}$ is universal for zero-dimensional Polish spaces, i.e., every such space is homeomorphic to a closed subspace of $\mathcal{N}$. It might be tempting to conjecture that $S_\infty$ is universal for the zero-dimensional Polish groups. This turns out to be false; for a counterexample see, e.g., Dougherty [94]. However, 1.5.1 (iv) shows that $S_\infty$ is universal for Polish groups, admitting a left-invariant ultrametric.
1. Polish Groups

From the preceding result it follows immediately that the class of closed subgroups of $S_\infty$ is (up to isomorphism) closed under quotients (by normal closed subgroups) and countable products. For instance, all Polish groups of the form $\prod_n H_n, H_n$ countable discrete, are (isomorphic to) closed subgroups of $S_\infty$.

Another characterization of the closed subgroups of $S_\infty$ comes from model theory.

Let $L = \{R_i\}_{i \in I}$ be a countable relational language and $A = (\mathbb{N}, \{R^A_i\}_{i \in I})$ a structure for $L$ with universe $\mathbb{N}$. Then

$$\text{Aut}(A),$$

the group of automorphisms of $A$, is a closed subgroup of $S_\infty$. Conversely, if $G \subseteq S_\infty$ is a subgroup of $S_\infty$ (not necessarily closed), let for each $n, I_n$ be the set of orbits of $G$ acting on $\mathbb{N}^n$, i.e., the equivalence classes of the following equivalence relation on $\mathbb{N}^n$,

$$s \sim_G t \iff \exists g \in G(g \circ s = t).$$

Then if $I = \bigcup_{n \geq 1} I_n$ and for each $i \in I_n$, we let $R_i$ be an $n$-ary relation symbol, consider the following canonical structure $A_G$ associated with $G$:

$$A_G = (\mathbb{N}, \{R^A_i\}_{i \in I}),$$

where $R^A_i = i \subseteq \mathbb{N}^n$. It is easy to check that $\text{Aut}(A_G) = \bar{G}$, so if $G$ is closed, then $G = \text{Aut}(A_G)$. It follows that the closed subgroups of $S_\infty$ are exactly the automorphism groups of countably infinite structures (with universe $\mathbb{N}$, in a countable relational language). For more on this, see, e.g., Cameron [90] or Hodges [93].

We conclude by mentioning some other interesting properties of the group $S_\infty$.

A Polish group is said to have the small index property if every subgroup of index $< 2^{\aleph_0}$ is open. If a closed subgroup $G$ of $S_\infty$ has the small index property, then the open subgroups of $G$ are exactly those subgroups of index $< 2^{\aleph_0}$, which means that the topology of $G$ is completely determined by its algebraic structure. It follows, for example, that if $G$, a closed subgroup of $S_\infty$, has the small index property, then any (algebraic) homomorphism $\varphi : G \to S_\infty$ is automatically continuous. A result of Rabinovich states that $S_\infty$ has the small index property. There are also many other known closed subgroups, of $S_\infty$, viewed as automorphism groups of various structures, that are known to have the small index property. For more on