

# 1

## Fermi gas

The basic properties of free electron systems are introduced. Then, the many-body effects of electron gas are discussed. The ground state energy is obtained by taking the screening effect into account.

### 1.1 Metals

Metals are composed of positive ions and conduction electrons making itinerant motion all over the crystal. A positive ion is composed of a nucleus and core electrons bounded around it. Conduction electrons lower the kinetic energy by making itinerant motion compared with the state bounded to a positive ion. This point is important in metallic cohesion.

For simple metals such as Na and Al, the Hamiltonian is given by

$$\mathcal{H} = \mathcal{H}_i + \mathcal{H}_e + \mathcal{H}_{e-i}, \quad (1.1)$$

$$\mathcal{H}_i = \sum_i \frac{\mathbf{P}_i^2}{2M} + \frac{1}{2} \sum_{i \neq j} V(\mathbf{R}_i - \mathbf{R}_j), \quad (1.2)$$

$$\mathcal{H}_e = \sum_i \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (1.3)$$

$$\mathcal{H}_{e-i} = \sum_{ij} v(\mathbf{r}_i - \mathbf{R}_j). \quad (1.4)$$

Here,  $\mathcal{H}_i$  in (1.2) represents the positive ion system; we assume one kind of positive ion with mass  $M$ .  $\mathbf{P}_i$  is the momentum of ion  $i$  and  $V(\mathbf{R}_i - \mathbf{R}_j)$  is the potential between ions, which depends only on their distance.  $\mathcal{H}_e$  in (1.3) is the Hamiltonian for the electron system with electron mass  $m$ ; the first and second terms denote the

kinetic energy and the coulomb interaction between electrons, respectively.  $\mathcal{H}_{e-i}$  in (1.4) represents the potential between electrons and positive ions. The Hamiltonian combined with these terms can describe various properties including magnetism and superconductivity. In this book we discuss mainly the second term of (1.3), electron interaction. Owing to the coulomb interaction, electrons move so as to avoid each other. This kind of correlated electron motion is called ‘electron correlation’.

When we are mainly interested in electron interaction, we simplify the positive ions by replacing them with a uniform positive charge distributed over the crystal and discuss only the term  $\mathcal{H}_e$ . By this replacement, we can avoid the difficulty arising from a periodic potential and lattice vibrations.

## 1.2 Free Fermi gas

An electron is a Fermi particle with a spin of  $1/2$ . There exist  $10^{22}$ – $10^{23}$  conduction electrons per cubic centimetre in most metals. For simplicity, let us ignore electron interactions among these, and also the periodic potential due to the positive ions, and consider the system composed of free electrons. We assume a system with  $N$  electrons in a cube of side  $L = \Omega^{1/3}$ . The wave-function  $\varphi_{\mathbf{k}}(\mathbf{r})$  for a free electron with wave-vector  $\mathbf{k}$  is given by

$$\varphi_{\mathbf{k}} = \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (1.5)$$

Here we take the periodic boundary condition

$$\varphi_{\mathbf{k}}(x + L, y, z) = \varphi_{\mathbf{k}}(x, y + L, z) = \varphi_{\mathbf{k}}(x, y, z + L) = \varphi_{\mathbf{k}}(x, y, z). \quad (1.6)$$

By substituting (1.5) into (1.6), we obtain

$$e^{ik_x L} = e^{ik_y L} = e^{ik_z L} = 1. \quad (1.7)$$

By this condition, values of  $\mathbf{k}$  are given by integers  $n_1, n_2$  and  $n_3$  as

$$k_x = 2\pi n_1/L, \quad k_y = 2\pi n_2/L, \quad k_z = 2\pi n_3/L. \quad (1.8)$$

Thus, wave-vector  $\mathbf{k}$  corresponds to a lattice point with unit of  $2\pi/L$  in the wave-vector space.

The energy  $\varepsilon_{\mathbf{k}}$  of a free electron with wave-vector  $\mathbf{k}$  is given by

$$\varepsilon_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} \left( \frac{2\pi}{L} \right)^2 (n_1^2 + n_2^2 + n_3^2). \quad (1.9)$$

Let us construct the ground state composed of  $N$  free electrons. Each one-electron state specified by wave-vector  $\mathbf{k}$  and spin quantum number  $\sigma$  can be occupied by only one electron because of the Pauli exclusion principle. In the ground state electrons occupy the  $N$  states from the lowest energy state to the  $N$ th lowest state.

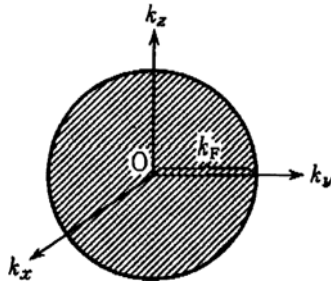


Fig. 1.1 Fermi sphere. Each  $\mathbf{k}$  point inside the sphere with radius  $k_F$  is occupied by one up-spin electron and one down-spin electron.

The highest occupied energy  $\varepsilon_F$  and wave-number  $k_F$  are called the Fermi energy and Fermi wave-number, respectively;  $\varepsilon_F$  and  $k_F$  satisfy the relation

$$\varepsilon_F = \frac{\hbar^2}{2m} k_F^2. \quad (1.10)$$

Thus, we have the ground state in which two electrons with up- and down-spins occupy the state  $\mathbf{k}$  within the sphere with radius  $k_F$ . Since the wave-vector  $\mathbf{k}$  is situated at a lattice point spaced  $2\pi/L$  apart, wave-vectors distribute in uniform density  $(L/2\pi)^3 = \Omega/(2\pi)^3$ . As a result, the electron number  $N$  is related to  $k_F$  by

$$N = \frac{2\Omega}{(2\pi)^3} \frac{4\pi}{3} k_F^3, \quad (1.11)$$

where factor 2 represents the degree of freedom arising from electron spin. From (1.11),  $k_F$  is given by electron density  $n = N/\Omega$  as

$$k_F = (3\pi^2 n)^{1/3}. \quad (1.12)$$

Let us calculate the number of electron states between energy  $E$  and  $E + \Delta E$ ,  $\Delta E$  being an infinitesimal energy. Assuming the total number of states below energy  $E$  as  $N(E)$ , we obtain

$$N(E + \Delta E) - N(E) = \frac{dN(E)}{dE} \Delta E. \quad (1.13)$$

Here,

$$dN(E)/dE = \rho(E) \quad (1.14)$$

is the energy density of states. Using (1.10) and (1.11), we obtain

$$\rho(E) = \frac{dN(k)}{dk} \frac{dk}{dE} = \frac{2\Omega}{(2\pi)^3} 4\pi k^2 \frac{m}{\hbar^2 k} = \frac{\Omega k m}{\pi^2 \hbar^2} = \frac{\Omega m}{\pi^2 \hbar^3} \sqrt{2mE}. \quad (1.15)$$

The density of states for a free electron system is proportional to  $\sqrt{E}$ .

Now let us consider real metals. We assume  $n = 10^{22} \text{ cm}^{-3}$  and obtain  $k_F \simeq 10^8 \text{ cm}^{-1}$  from (1.12). The value of  $1/k_F$  is around  $1 \text{ \AA} = 10^{-8} \text{ cm}$ , corresponding to atomic distance. Inserting this value into (1.10) and using  $\hbar = 1 \times 10^{-27} \text{ erg s}$  and  $m = 9 \times 10^{-28} \text{ g}$ , we obtain  $\varepsilon_F \simeq 6 \times 10^{-12} \text{ erg} \simeq 4 \text{ eV}$ . Since 1 eV is the energy corresponding to  $10^4 \text{ K}$ , the room temperature, 300 K, is sufficiently low compared with the Fermi temperature,  $T_F = \varepsilon_F/k_B$ . This fact is very important in understanding the electronic specific heat discussed in the next section.

The level splitting of one-electron energy near the Fermi surface is given by  $\Delta\varepsilon = \varepsilon_F/N \simeq 10^{-22} \text{ eV} \simeq 10^{-18} k_B \text{ K}$ . As a result, electron–hole pair excitations near the Fermi energy can be created with vanishingly small excitation energy. These electron–hole pair excitations exist in an infinite number. Thus, the conduction electron system in the Fermi degeneracy is degenerate in the infinite number of states. In general, the degenerated states suffer a strong effect even under small perturbations. The special nature of the Fermi surface, which can be called ‘fragility’, plays an important role in the orthogonality theorem and the theory of superconductivity.

### 1.3 Electronic specific heat and Pauli susceptibility

If we apply the principle of equipartition to the free electron system to calculate the specific heat, we obtain the following result. The internal energy  $W$  is given by

$$W = \frac{3}{2} N k_B T, \quad (1.16)$$

$k_B$  being the Boltzmann constant. The specific heat  $C_V$  takes a constant value,

$$C_V = \frac{dW}{dT} = \frac{3}{2} N k_B. \quad (1.17)$$

This value is expected to give the same order of contributions as the lattice specific heat around room temperature. However, in actual metals we cannot observe such a large electronic specific heat at room temperature. This is because the room temperature is too low compared with the Fermi temperature for the principle of equipartition to be applicable. Only the electrons limited within the narrow width of temperature in the vicinity of the Fermi energy contribute to the specific heat.

Now, let us calculate correctly the specific heat due to free electrons. Using the chemical potential  $\mu$ , the distribution of electrons is given by the Fermi distribution function

$$f(\varepsilon_k) = \left[ 1 + \exp\left(\frac{\varepsilon_k - \mu}{k_B T}\right) \right]^{-1}. \quad (1.18)$$

In this case the internal energy of the electron system  $W$  is given by

$$W = 2 \sum_k \varepsilon_k f(\varepsilon_k), \quad (1.19)$$

where the factor 2 arises from spin degeneracy. Using the density of states  $\rho(\varepsilon_k)$  including both spins, we obtain

$$W = \int_0^\infty \varepsilon \rho(\varepsilon) f(\varepsilon) d\varepsilon. \quad (1.20)$$

The total number of electrons  $N$  is given by

$$N = \sum_{k\sigma} f(\varepsilon_k) = \int_0^\infty \rho(\varepsilon) f(\varepsilon) d\varepsilon. \quad (1.21)$$

This equation at  $T = 0$  becomes

$$N = \int_0^{\varepsilon_F} \rho(\varepsilon) d\varepsilon. \quad (1.22)$$

Now let us consider the following integral  $I$  to accomplish the calculation at low temperatures:

$$I = \int_0^\infty g(\varepsilon) f(\varepsilon) d\varepsilon. \quad (1.23)$$

Here,  $g(\varepsilon)$  is a smooth function of energy  $\varepsilon$ . Using a partial integration, we obtain

$$I = [G(\varepsilon) f(\varepsilon)]_0^\infty - \int_0^\infty G(\varepsilon) \frac{\partial f}{\partial \varepsilon} d\varepsilon, \quad (1.24)$$

$$G(\varepsilon) = \int_0^\varepsilon g(\varepsilon) d\varepsilon. \quad (1.25)$$

The first term of (1.24) vanishes because  $f(\infty) = 0$ . To calculate the second term, we expand  $G(\varepsilon)$  around  $\varepsilon = \mu$ . Writing the  $n$ th derivative of  $G$  as  $G^{(n)}$ , we obtain

$$G(\varepsilon) = G(\mu) + (\varepsilon - \mu)G'(\mu) + \frac{1}{2}(\varepsilon - \mu)^2 G''(\mu) + \dots \quad (1.26)$$

Inserting this into (1.24), we get

$$\begin{aligned} I &= G(\mu) \int_0^\infty \left(-\frac{\partial f}{\partial \varepsilon}\right) d\varepsilon + G'(\mu) \int_0^\infty (\varepsilon - \mu) \left(-\frac{\partial f}{\partial \varepsilon}\right) d\varepsilon + \dots \\ &+ \frac{G^{(n)}(\mu)}{n!} \int_0^\infty (\varepsilon - \mu)^n \left(-\frac{\partial f}{\partial \varepsilon}\right) d\varepsilon + \dots \end{aligned} \quad (1.27)$$

The first term gives  $G(\mu)$ . The general terms are given by

$$\begin{aligned} \frac{1}{n!} \int_0^\infty (\varepsilon - \mu)^n \left(-\frac{\partial f}{\partial \varepsilon}\right) d\varepsilon &= \frac{(k_B T)^n}{n!} \int_{-\infty}^\infty \frac{z^n}{(e^z + 1)(1 + e^{-z})} dz \\ &= \begin{cases} 2c_n (k_B T)^n & (n \text{ even}) \\ 0 & (n \text{ odd}). \end{cases} \end{aligned} \quad (1.28)$$

As an example, for  $n = 2$ ,

$$2c_2 = \frac{1}{2} \int_{-\infty}^\infty \frac{z^2 dz}{(e^z + 1)(1 + e^{-z})} = \frac{\pi^2}{6}. \quad (1.29)$$

As a result,  $I$  is given by

$$I = \int_0^\mu g(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 \left[ \frac{\partial g(\varepsilon)}{\partial \varepsilon} \right]_{\varepsilon=\mu} + \dots \quad (1.30)$$

Applying this result to (1.20) and (1.21), we obtain

$$W = \int_0^\mu \varepsilon \rho(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 \left[ \frac{\partial}{\partial \varepsilon} (\varepsilon \rho(\varepsilon)) \right]_{\varepsilon=\mu} + \dots \quad (1.31)$$

$$\begin{aligned} C_V &= \frac{dW}{dT} = \mu \rho(\mu) \frac{d\mu}{dT} + \frac{\pi^2}{3} k_B^2 T \left[ \rho(\varepsilon) + \mu \frac{\partial \rho}{\partial \varepsilon} \right]_{\varepsilon=\mu} + O(T^2) \\ &= \frac{\pi^2}{3} k_B^2 \rho(\varepsilon_F) T + \mu \rho(\mu) \left[ \frac{d\mu}{dT} + \frac{\pi^2 k_B^2 T}{3 \rho(\mu)} \frac{\partial \rho}{\partial \varepsilon} \right]_{\varepsilon=\mu} \\ &= \frac{\pi^2}{3} k_B^2 \rho(\varepsilon_F) T. \end{aligned} \quad (1.32)$$

In (1.32) we have used the shift of chemical potential

$$\mu = \varepsilon_F - \frac{\pi^2}{6} (k_B T)^2 \left[ \frac{\partial}{\partial \varepsilon} \log \rho(\varepsilon) \right]_{\varepsilon=\mu}, \quad (1.33)$$

which is obtained by substituting  $g(\varepsilon) = \rho(\varepsilon)$  in (1.30) and shifting  $\mu$  so as to conserve the total electron number  $N$ .

Thus, the electronic specific heat at low temperatures is proportional to the density of states  $\rho(\varepsilon_F)$  on the Fermi surface and given by the  $T$ -linear term as shown in (1.32). Equation (1.32) can be written as

$$\begin{aligned} C_V &= \gamma T, \\ \gamma &= \frac{\pi^2}{3} k_B^2 \rho(\varepsilon_F). \end{aligned} \quad (1.34)$$

Here, the coefficient of the specific heat  $\gamma$  is called the Sommerfeld constant.

Let us apply a weak magnetic field  $H$  to the free electron system at low temperature and obtain the expression for the Pauli susceptibility. Using a  $g$ -value of  $g = 2$  and the Bohr magneton  $\mu_B$ , the Zeeman energy with spin  $\sigma$  is given by  $g\sigma\mu_B H/2 = \sigma\mu_B H$ . The Zeeman energy induces the magnetization given by  $\Delta M = \mu_B(\delta n_\downarrow - \delta n_\uparrow) = \mu_B^2 \rho(\varepsilon_F)H$ . Thus, the magnetic susceptibility  $\chi$  is given by

$$\chi = \Delta M/H = \mu_B^2 \rho(\varepsilon_F). \quad (1.35)$$

The Pauli susceptibility is proportional to the density of states  $\rho(\varepsilon_F)$  on the Fermi surface; this is common to the coefficient of specific heat  $\gamma$ .

#### 1.4 Many-body effect of electron gas

The effects of coulomb interaction on electron gas had been made clear in the 1950s by the efforts of many people, such as Bohm, Pines, Nozières, Gell-Mann, Brueckner and Sawada. As shown below, initially, by a straight perturbation calculation, the essential points of the problem were made clear. Then, the difficult key problems were solved by physical considerations, creating new concepts and developing the methods of the many-body problem. Since the history of the investigations on electron gas is instructive, we describe it here in detail as a starting point for the many-body problem [1–6].

In order to discuss mainly the effect of electron interaction, we assume the electron gas model in which the positive charge due to ions is replaced by a uniform one. As a result, in this model the uniform distribution of electrons in space cancels with the positive background to give no effect. The deviation from the uniform distribution gives rise to the coulomb interaction among electrons. The Hamiltonian of this system is given by  $\mathcal{H}_e$  in (1.3) and is written as

$$\mathcal{H}_e = \sum_i \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (1.36)$$

where  $\mathbf{r}_i$  and  $\mathbf{p}_i$  are the position and momentum of electron  $i$ , respectively. Here we define electron density  $\rho(\mathbf{r})$  and its Fourier transform  $\rho_q$ :

$$\rho(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i) = \sum_q \rho_q e^{i\mathbf{q} \cdot \mathbf{r}}, \quad (1.37)$$

$$\rho_q = \frac{1}{\Omega} \sum_i e^{-i\mathbf{q} \cdot \mathbf{r}_i} = \rho_{-q}^\dagger. \quad (1.38)$$

The  $\mathbf{q} = 0$  component of  $\rho_{\mathbf{q}}$ ,  $\rho_0 = N_e/\Omega = n$ , is the average electron density and cancels with the uniform positive charge. Using  $\rho_{\mathbf{q}}$ , we can write (1.36) as

$$\mathcal{H}_e = \sum_i \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{\mathbf{q}} V_{\mathbf{q}} (\Omega \rho_{\mathbf{q}}^\dagger \rho_{\mathbf{q}} - n), \quad (1.39)$$

where  $V_{\mathbf{q}}$  is the Fourier transform of the coulomb interaction  $e^2/r$ :

$$V_{\mathbf{q}} = \frac{4\pi e^2}{q^2}. \quad (1.40)$$

Here it is noted that since

$$\rho_{\mathbf{q}}^\dagger \rho_{\mathbf{q}} = \frac{1}{\Omega^2} \sum_{ij} e^{i\mathbf{q}\cdot(\mathbf{r}_i - \mathbf{r}_j)}, \quad (1.41)$$

the term  $i = j$  gives  $n/\Omega$  and (1.39) excludes the coulomb interaction with  $\mathbf{r}_i = \mathbf{r}_j$ , in agreement with (1.36).

To treat the coulomb interaction as a perturbation, let us represent (1.36) in the second quantization form (see Appendix B). The coulomb integral is written by the wave-functions  $\varphi_{\sigma_1}(\mathbf{r}_1)$  and  $\varphi_{\sigma_2}(\mathbf{r}_2)$ , as

$$\iint d\mathbf{r}_1 d\mathbf{r}_2 \varphi_{\sigma_1}^*(\mathbf{r}_1) \varphi_{\sigma_2}^*(\mathbf{r}_2) \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|} \varphi_{\sigma_2}(\mathbf{r}_2) \varphi_{\sigma_1}(\mathbf{r}_1). \quad (1.42)$$

By expanding the wave-function  $\varphi_{\sigma}(\mathbf{r})$  with the plane waves as

$$\varphi_{\sigma}(\mathbf{r}) = \sum_{\mathbf{k}} a_{\mathbf{k}\sigma} \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (1.43)$$

we write the coulomb interaction as

$$\mathcal{H}_C = \frac{1}{\Omega} \sum_{\mathbf{q} \neq 0} \frac{2\pi e^2}{q^2} \sum_{\substack{\mathbf{k}_1 \mathbf{k}_2 \\ \sigma_1 \sigma_2}} a_{\mathbf{k}_1 + \mathbf{q} \sigma_1}^\dagger a_{\mathbf{k}_2 - \mathbf{q} \sigma_2}^\dagger a_{\mathbf{k}_2 \sigma_2} a_{\mathbf{k}_1 \sigma_1}. \quad (1.44)$$

Here  $a_{\mathbf{k}\sigma}$  ( $a_{\mathbf{k}\sigma}^\dagger$ ) is the annihilation (creation) operator of the electron with wave-vector  $\mathbf{k}$  and spin  $\sigma$ . The summation over  $\mathbf{q}$  in (1.44) excludes the part of  $\mathbf{q} = 0$  because it cancels with the positive charge.

The density fluctuation  $\rho_{\mathbf{q}}$  is given by

$$\rho_{\mathbf{q}} = \frac{1}{\Omega} \sum_{\mathbf{k}\sigma} a_{\mathbf{k} - \mathbf{q}\sigma}^\dagger a_{\mathbf{k}\sigma}. \quad (1.45)$$



Thus,  $\mathcal{H}_e$  is given in the second quantization as

$$\mathcal{H}_e = \sum_{k\sigma} \varepsilon_k a_{k\sigma}^\dagger a_{k\sigma} + \frac{1}{2\Omega} \sum_{\substack{kk', q \neq 0 \\ \sigma\sigma'}} V_q a_{k+q\sigma}^\dagger a_{k'-q\sigma'}^\dagger a_{k'\sigma'} a_{k\sigma}, \quad (1.46)$$

where  $\varepsilon_k$  is given by

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}. \quad (1.47)$$

Operators  $a_{k\sigma}^\dagger$  and  $a_{k\sigma}$  satisfy the commutation rule for Fermi particles (see Appendix B):

$$\begin{aligned} [a_{k\sigma}, a_{k'\sigma'}]_+ &= [a_{k\sigma}^\dagger, a_{k'\sigma'}^\dagger]_+ = 0, \\ [a_{k\sigma}, a_{k'\sigma'}^\dagger]_+ &= \delta_{k,k'} \delta_{\sigma,\sigma'}. \end{aligned} \quad (1.48)$$

Now we study the effect of the coulomb interaction using the above results. First of all, let us calculate the ground state energy by regarding the second term, the coulomb interaction, as a perturbation on the first term, the kinetic energy. The unperturbed state given by the first term is the Fermi sphere occupied up to  $k_F$  by two electrons with up- and down-spin. We write it as  $|0\rangle$  and obtain

$$n_{k\sigma} = \langle 0 | a_{k\sigma}^\dagger a_{k\sigma} | 0 \rangle = \begin{cases} 1 & (k < k_F) \\ 0 & (k > k_F). \end{cases} \quad (1.49)$$

The average kinetic energy per electron in the unperturbed state is given by

$$\varepsilon_{\text{kin}} = \frac{1}{N_e} \langle 0 | \sum_{k\sigma} \varepsilon_k a_{k\sigma}^\dagger a_{k\sigma} | 0 \rangle = \frac{3}{5} \varepsilon_F, \quad (1.50)$$

where  $\varepsilon_F$  is the Fermi energy given by (1.10). The result of (1.50) is obtained from

$$\varepsilon_{\text{kin}} = \int_0^{k_F} \frac{\hbar^2 k^2}{2m} 4\pi k^2 dk \bigg/ \int_0^{k_F} 4\pi k^2 dk. \quad (1.51)$$

Now let us introduce  $r_0$  representing electron density as

$$\frac{\Omega}{N_e} = \frac{1}{n} = \frac{4\pi}{3} r_0^3. \quad (1.52)$$

The radius  $r_0$  of the sphere for one electron is replaced by the dimensionless parameter  $r_s$ , which is given in units of Bohr radius as

$$r_s = r_0/a_B. \quad (1.53)$$

The Fermi energy  $\varepsilon_F$  is given by (1.12) as

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3} = \left(\frac{9\pi}{4}\right)^{2/3} \frac{1}{r_s^2} \text{Ry}. \quad (1.54)$$

The unit of energy Ry corresponds to the ionization energy of atomic hydrogen:

$$1 \text{ Ry} = \frac{me^4}{2\hbar^2} = \frac{e^2}{2a_B} \simeq 13.5 \text{ eV} = 2.17 \times 10^{-11} \text{ erg}. \quad (1.55)$$

As a final result, the average kinetic energy of (1.50) is given by

$$\varepsilon_{\text{kin}} = \frac{3}{5} \varepsilon_F \simeq \frac{2.21}{r_s^2} \text{ Ry}. \quad (1.56)$$

### 1.5 Exchange energy

The first-order perturbation term with respect to the coulomb interaction in the ground state energy is given by

$$\begin{aligned} E_1 &= \frac{1}{\Omega} \sum_{\substack{kk' \\ q\sigma}} \frac{V_q}{2} \langle 0 | a_{k+q\sigma}^\dagger a_{k'-q\sigma'}^\dagger a_{k'\sigma} a_{k\sigma} | 0 \rangle \\ &= \frac{1}{\Omega} \left\{ \sum_{\substack{kk' \\ \sigma\sigma'}} \frac{V_{q=0}}{2} n_{k\sigma} n_{k'\sigma'} + \sum_{\substack{kq \\ \sigma}} -\frac{V_q}{2} n_{k+q\sigma} n_{k\sigma} \right\}. \end{aligned} \quad (1.57)$$

The first term corresponding to  $q = 0$  in (1.57) cancels with the positive charge of the background. The second term with  $q = k' - k$ , to which only the electrons possessing parallel spins contribute, arises from the exchange integral of the coulomb interaction and possesses a negative sign. The term denoted as  $E_{\text{ex}}$  is given by

$$\begin{aligned} E_{\text{ex}} &= -\frac{1}{\Omega} \sum_{\substack{kq \\ \sigma}} \frac{V_q}{2} n_{k+q\sigma} n_{k\sigma} = \frac{2}{\Omega} \sum_{\substack{k_1 < k_F \\ k_2 < k_F}} \frac{-2\pi e^2}{|\mathbf{k}_1 - \mathbf{k}_2|^2} \\ &= -\sum_{k_1 \leq k_F} \frac{e^2}{2\pi} \left\{ \left( \frac{k_F^2 - k_1^2}{2k_1} \right) \log \left| \frac{k_F + k_1}{k_F - k_1} \right| + k_F \right\}. \end{aligned} \quad (1.58)$$

Then, by integrating this over  $k_1$ ,

$$E_{\text{ex}} = -2 \frac{\Omega}{(2\pi)^3} e^2 k_F^4. \quad (1.59)$$