AN INTRODUCTION TO GRANULAR FLOW

The flow of granular materials such as sand, snow, coal, and catalyst particles is a common occurrence in natural and industrial settings. They are important since a large fraction of the materials handled and processed in the chemical, metallurgical, pharmaceutical, and food-processing industries are granular in nature. The mechanics of these materials' flows is not well understood. This book describes the theories for granular flow based mainly on continuum models, although alternative discrete models are also discussed briefly. The level is appropriate for advanced undergraduates or beginning graduate students. The goal is to inform the reader about observed phenomena and some available models and their shortcomings and to visit some issues that remain unresolved. There is a selection of problems at the end of the chapters to encourage exploration, and extensive references are given.

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To our teachers,

Professors M. S. Ananth and Roy Jackson

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Preface

The flow of granular materials such as sand, snow, coal, and catalyst particles is a common occurrence in natural and industrial settings. Unfortunately, the mechanics of these materials is not well understood. Experiments reveal complex and, at times, unexpected behavior, whereas existing theories are often tentative and do not represent the entire range of observed behavior. Nevertheless, significant advances have been made in the understanding of the mechanics of granular flows, and the time is ripe for an account of experimental observations and theoretical models pertaining to flow in relatively simple geometries.

The importance of understanding granular flows need not be overstated – a large fraction of the materials handled and processed in the chemical, metallurgical, pharmaceutical, and food-processing industries are granular in nature. The flow and transportation of these materials are often critical operations in these processes. In most cases, the design of processes and equipment is based largely on experience and empirical rules. An appreciation of the underlying principles may be helpful in developing better design and operating procedures.

Some of the early investigations of granular flow were motivated by the need to understand the deformation of soils subjected to external loads, such as large structures. The deformation rates in these processes are usually very small. Theoretical models for these slow flows have increased in sophistication and complexity over the years, borrowing concepts from metal plasticity and soil mechanics. A contrasting picture of granular flow has emerged over the last three decades. This is believed to be applicable to rapid flows, where the deformation rates are large. Models for rapid flows have been based mainly on the kinetic theory of dense gases, with suitable modifications to account for the inelasticity of interparticle collisions and particle roughness. These models assume that momentum transfer occurs by collisions of short duration between particles and by free flight of particles between collisions. In contrast, momentum transfer during slow flow occurs via contacts of a longer duration between particles that slide and roll relative to each other. A realistic picture of particle interactions that encompasses these extremes is not available at present, barring some tentative attempts.

The theories described here for granular flow are based mainly on continuum models, although alternative discrete models are also discussed very briefly. Chapter 1 describes the qualitative behavior of granular materials in various situations and formulates balance laws. Chapter 2 discusses constitutive equations for slow, plane flow. Chapters 3 and 4 deal with flow in wedge-shaped hoppers and bunkers, respectively; after summarizing some of the experimental observations, the equations formulated in Chapter 2 are used to construct approximate solutions. Chapter 5 deals with constitutive equations for slow three-dimensional flow; these are applied to flow through axisymmetric hoppers and bunkers in Chapter 6. Using kinetic theory, constitutive equations for rapid granular flow are formulated in Chapter 7 and applied to flow between parallel plates and down inclined chutes in Chapter 8. The kinetic theory is extended to accommodate particle roughness in Chapter 9. Finally, a few tentative attempts to model flow in the intermediate regime, where a range of interparticle contact times is present, are desribed in Chapter 10. For the reader interested in details, extensive references are given. Some problems are included at the end of most of the chapters, to enable the reader to either fill in details that are omitted in the book or extend the material.

Overall, we wish to inform the reader about observed phenomena, some of the available models and their shortcomings, and unresolved issues in granular flow. We describe in detail only a very small number of "animals" in the vast and rapidly expanding granular zoo, but we hope that the material presented here will stimulate readers to learn more about this field. Many interesting phenomena and puzzles lurk in topics such as vibrated beds, segregation in rotating cylinders, landslides and sand dunes, the behavior of cohesive powders, and fluid–particle flows, which either have not been discussed here or are mentioned only in passing.

This book deals with granular flow at a level that should be suitable for senior undergraduate and postgraduate students. A knowledge of undergraduate-level fluid and solid mechanics is desirable but not essential. As the book provides a reasonably self-contained introduction to the subject, it may also be useful to new entrants to the field. It is hoped that a study of the simple problems discussed here may prepare the reader for a better understanding of more realistic and complex situations, which are often encountered in industries. Parts of this book are based on graduate courses taught at Princeton University, Indian Institute of Science, and California Institute of Technology.

We request readers to inform us by email, (kesava@chemeng.iisc.ernet.in or prnott@chemeng.iisc.ernet.in) if they come across errors. We shall also be grateful for suggestions that may improve the book. An errata for the book and supplementary information, such as solutions to problems (for instructors) and additional problems, may be found at the catalogue page of the book (http://www.cambridge.org/us/catalogue/catalogue.asp?isbn=0521571669).

We are grateful to our teacher Prof. R. Jackson for introducing us to this field and to Dr. R. M. Nedderman for prodding one of us to write a book on it. Prof. S. Sundaresan had originally planned to be one of the coauthors. He opted out later, but we are very grateful to him for his comments and suggestions on some of the chapters. We are grateful to professors B. Ananthanarayan, I. Goldhirsch, R. Jackson, C. S. Jog, H. S. Mani, S. Ramaswamy, and D. Sen for helpful discussions. We express our heartfelt thanks to our editors, Ms. Florence Padgett, Mr. Roger Astley, and Mr. Peter Gordon, who showed extraordinary patience in dealing with our repeated requests for extensions. We are grateful to Ms. B. G. Girija, Mr. K. Venugopal, Mr. Gautam Parthasarathy, and Mr. Vishwajeet Mehandia for drawing some of the figures; Mr. P. T. Raghuram, Ms. Shruti Seshadri, Mr. Alok Srivastava, Dr. S. Venugopal, and Prof. Sanjeev Gupta for assistance with photographs; the staff of Aptara for promptly responding to our queries regarding LaTeX, and Mr. Anoop Chaturvedi of Aptara for responding to many queries during our review of the copyedited manuscript. Special thanks are due to various publishers and societies who have permitted us to reproduce figures from their journals, to professors A. Drescher and R. P. Behringer for permitting us to use some of their photographs, and to Prof. D. W. Agar for translating the titles of some articles from French and German to English. Finally, we wish to record our gratitude to and deep appreciation for our families, who tolerated our preoccupation with this work for an extended period of time.

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Notation

a_1–a_6	constants of O(1) in the heuristic high-density theory for rapid flows, §7.2 and 8.1.1
a_n, b_n	coefficients in the expansion of $\mathcal{A}$, $\mathcal{B}$ in (7.126)
$\mathbf{b}$	body force per unit mass
$\hat{\mathbf{b}}$	scaled body force per unit mass, defined in (7.79)
b_{*0}	$\equiv b_*(\theta = 0)$, constant occurring in the boundary conditions for the radial stress field (see (6.32) and (6.48))
c	cohesion
c_w	adhesion
c_1, c_2, c_3	principal compressive rates of deformation, i.e., the eigenvalues of $\mathbf{C}$
c_*	$= (c_1 + c_2)/2$
$\mathbf{c}$	translation velocity of particles
$\hat{\mathbf{c}}$	scaled translation velocity, defined in (7.79)
$\mathbf{c}'$	postcollisional translation velocity of a particle
$\mathbf{c}''$	precollisional translation velocity of a particle in an "inverse collision," as defined in §7.3.2
d_p	particle diameter
d_w	diameter of a wall hemisphere (see Fig. 8.2)
$\overline{d}$	$\equiv \frac{1}{2}(d_\mathrm{p} + d_\mathrm{w})$
det	determinant of a tensor or a matrix, defined in (A.37) and (A.41)
e_p	coefficient of restitution for particle collisions, defined in (7.3)
e_w	coefficient of restitution for particle–wall collisions
$\mathbf{e}_i$	basis vector for a Cartesian coordinate system
f, $f^{(1)}$	single-particle probability distribution function of particle position and velocity, sometimes referred to as the singlet distribution
f^0	Maxwell–Boltzmann distribution based on the local density, velocity, and temperature
$\hat{f}$	scaled singlet distribution, defined in (7.79)
f'	singlet distribution for the postcollision velocity of a "direct collision" (see §7.3.2)
f''	singlet distribution for the precollision velocity of an "inverse collision"
$f^{(2)}$	probability distribution function of the position and velocity of a pair of particles
$\dot{f}_\mathrm{coll}$	rate of change of $f^{(1)}$ due to collisions
f_d	drag force exerted by the fluid on a single particle
g	acceleration due to gravity; $\equiv n^{(2)}/n^2$, pair distribution function

$\mathbf{g}$	translation velocity of particle 1 relative to particle 2
$\hat{\mathbf{g}}$	$\equiv \mathbf{g}/v_s$, scaled relative translation velocity
$\mathbf{g}'$	postcollisional value of $\mathbf{g}$
$g_0(\theta)$	function defining the radial stress field
$g_0(\nu)$	equilibrium pair distribution function at contact (Chapters 7–10)
$g_w(\nu)$	enhancement in the number density of particles in contact with the wall, defined in §8.1.2
$\mathbf{h}$	relative velocity of the point of contact of particle 1 with respect to that of particle 2, given by (9.1)
k_x, k_y, k_z	wavenumbers for the x, y, and z directions (§8.4)
$\mathbf{k}$	unit vector along the line joining the center of particle 1 to the center of particle 2 (see Fig. 7.3);
	unit vector along the line joining the center of a wall hemisphere to the center of a colliding particle (see Fig. 8.2)
$\mathbf{k}''$	$\mathbf{k}$ for an "inverse collision" (see Fig. 7.6)
ℓ	$\equiv (1-e_w^2)\,a_6/a_3$, constant introduced in §8.2.1 and §8.3.3
m	$\equiv a_1^2/(a_3\,a_5\varphi)$, constant introduced in §8.2.1 and §8.3.3
m_i	mass of particle i
m_p	mass of particle
$\dot{m}$	mass flow rate per unit width in chute flow (§8.3 and 10.2)
$\dot{m}^*$	dimensionless mass flow rate, defined in (8.81)
m_ℓ	loading, that is, mass of material per unit area of the chute base (§8.3.1 and 10.2)
m_ℓ^*	dimensionless loading, defined in (8.81)
n	number density of particles
$n^{(2)}$	probability distribution function of the positions of a pair of particles
$\mathbf{n}$	unit normal to a bounding surface
p	mean stress, defined in (5.6); pressure in rapid granular flow
p_c	mean stress at a critical state
p_f	fluid pressure
q	invariant of the stress tensor, defined in (5.6)
$\mathbf{q}$	pseudothermal energy flux;
	energy flux (Chapter 1)
$\mathbf{q}^s$, $\mathbf{q}^c$	streaming and collisional contributions to the pseudothermal energy flux
$\hat{\mathbf{q}}$	$\equiv \mathbf{q}/(\rho_p v_s^3)$, scaled pseudothermal energy flux (Chapter 7)
$\mathbf{q}_{pt}$	pseudothermal energy flux; referred to as $\mathbf{q}$ after (7.11)
$\mathbf{q}_r$	flux of rotational fluctuational kinetic energy, defined in (9.29)
$\mathbf{q}_w$	flux of fluctuational kinetic energy to the wall due to particle–wall collisions
$\mathbf{q}'$	flux of true thermal energy
r	radial coordinate measured from the virtual apex of the hopper (see Fig. 3.13);
	$\equiv \mathcal{N}/\mathcal{S}$, stress ratio introduced in §8.2.1
r_e	radial coordinate corresponding to the edge of the exit slot
$\mathbf{r}$	position vector of particle
$\hat{\mathbf{r}}$	scaled position vector, defined (7.79)
s	$= (c_1 - c_2)/2$, the magnitude of the maximum shear rate in plane flow;
	mean free path of particles (Chapters 7 and 8)
s_j	dimensionless arc length measured along the jth characteristic
$\bar{s}_w$	mean spacing between wall hemispheres (see Fig. 8.2)
s_{ij}	deviatoric stress component, $= \sigma_{ij} - p\delta_{ij}$
$\mathbf{s}$	$\equiv \hat{\boldsymbol{\Omega}}/(2\hat{T})^{1/2}$, rescaled peculiar spin (Chapter 9)
$\mathbf{s}'$	deviatoric stress tensor, $= \boldsymbol{\sigma} - p\,\mathbf{I}$

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Notation

$\hat{t}$	scaled time, defined in (7.79)
$\mathbf{t}_{(\mathbf{n})}$	stress vector, = force per unit area exerted on a surface with unit inward normal $\mathbf{n}$
tr	trace of a tensor, defined by (A.18)
u_w	pore pressure or pressure in excess of the atmospheric pressure which is exerted by the fluid on the particles
v_s	velocity scale, used to scale variables in (7.79)
$v*$	dimensionless velocity, defined in (8.71) for plane Couette flow and (8.104) for chute flow
$\mathbf{v}$	velocity of a material point, or mean velocity $\langle \mathbf{c} \rangle$ of a collection of particles
$\mathbf{v}_i$	velocity of particle i
$\hat{\mathbf{v}}$	$\equiv \mathbf{v}/v_\mathrm{s}$, scaled mean velocity
$\mathbf{v}_\mathrm{w}$	velocity of a wall
$\mathbf{v}_\mathrm{slip}$	$\equiv \mathbf{v} - \mathbf{v}_\mathrm{w}$, velocity slip at a wall
w	root-mean-square fluctuation velocity of particle (Chapters 7 and 8)
$\mathbf{w}$	local averaged velocity of the fluid phase relative to the particle phase
$y*$	dimensionless distance, defined in (8.71) for plane Couette flow and (8.104) for chute flow
A	dimensionless cross-sectional area, scaled by W^2
$\mathcal{A}, \mathcal{B}, \mathcal{R}$	functions of ξ in the expression for Φ_K in (7.125) and (9.70)
C	cohesion (Section 2.2); rate of compression in the direction of the normal to a plane, defined in (2.74)
C_{ij}	component of the rate of deformation tensor $\mathbf{C}$
$\mathbf{C}$	rate of deformation tensor, defined in the compressive sense by (2.49); $\equiv \mathbf{c} - \mathbf{v}$, peculiar velocity of a particle
$\hat{\mathbf{C}}$	$\equiv \mathbf{C}/v_\mathrm{s}$, scaled peculiar velocity of particles
$\mathrm{D}/\mathrm{D}t$	material derivative, defined by (1.23)
D	either the diameter or the width of the exit slot of a hopper or bunker (see Fig. 3.1)
$\mathbf{D}$	$= -\mathbf{C}$, rate of deformation tensor, defined by (1.50)
$\hat{\mathbf{D}}$	$\equiv \mathbf{D}H/v_\mathrm{s}$, scaled rate of deformation tensor
$\mathbf{F}_d$	drag force per unit volume of the suspension, exerted by the fluid phase on the particle phase
$\hat{\mathbf{G}}$	$\equiv (\hat{\mathbf{C}}_1 + \hat{\mathbf{C}})/2$, scaled mean peculiar velocity of colliding particles
H	head or height of the granular material above the exit slot of a hopper or bunker (see Fig. 3.2); characteristic macroscopic length scale in the Chapman–Enskog expansion (Chapters 7 and 9); distance between plates in plane Couette flow; thickness of flowing layer in chute flow
H^*	$\equiv H/d_\mathrm{p}$, dimensionless Couette gap in plane Couette flow, §8.2.1; dimensionless flow depth in chute flow, §8.3
H^*_hd	value of H^* for chute flow predicted by the high-density theory at a given angle of inclination θ
I	moment of inertia of a particle (Chapter 9)
I_1, I_2, I_3	principal invariants of the stress tensor, defined in (5.2)
$\hat{I}$	$\equiv 4I/(m_\mathrm{p} d_\mathrm{p}^2)$, dimensionless moment of inertia of a particle (Chapter 9)
$\mathbf{I}$	unit (identity) tensor
J_2	second invariant of the deviatoric stress tensor, defined by (5.9)
$\mathcal{J}$	Jacobian for the transformation of one set of variables to another

$\mathbf{J}$	impulse per unit mass on particle 2 during collision with particle 1
K	coefficient of earth pressure at rest, i.e., the ratio of the horizontal normal stress to the vertical normal stress;
	$\equiv Kn^{-1}$, inverse Knudsen number (Chapters 7–10)
$\mathcal{K}(\nu)$	$\equiv \kappa/(\rho_p d_p \sqrt{T})$, thermal conductivity function (Chapters 8 and 10)
Kn	$\equiv H/s$, Knudsen number, ratio of macroscopic to microscopic length scales
$\dot{M}$	mass flow rate
L	$\equiv a_4(1 - e_p^2)/a_3$, constant introduced in §8.2.1 and §8.3.3
$L_n^{(p)}(x)$	Generalized Laguerre polynomials, or Sonine polynomials, used in §7.3.10
$\mathcal{L}$	linearized Boltzmann operator, defined in (7.113)
$\mathcal{L}\mathrm{r}$	linearized Boltzmann operator for rough particles, defined in (9.47)
$\mathcal{L}^*$	linearized operator defined in (9.74)
M	$\equiv a_1^2/(a_2\, a_3)$, constant introduced in §8.2.1 and §8.3.3
$\mathbf{M}$	couple stress, defined in (9.27)
N	normal stress
N_b	normal stress exerted on the bin wall
N_h	normal stress exerted on the hopper wall
N_w	normal stress exerted on the wall of a bin or hopper
$\mathcal{N}$	normal stress on the walls in plane Couette flow (see Fig. 8.4)
P	dimensionless perimeter of the bin section, scaled by W
$\mathcal{P}$	$\equiv p/(\rho_p T)$, pressure function
$\mathbf{Q}$	orthogonal tensor, defined in (5.60)
R	particle radius
Re	Reynolds number
S	rate of shear in the direction of a unit vector $\mathbf{t}$ which is tangential to a surface, defined in (2.74)
S_m	bounding surface of a material volume
$\mathcal{S}$	shear stress on the walls in plane Couette flow (see Fig. 8.4)
$\boldsymbol{\mathcal{S}}_{\mathrm{w}}$	stress transmitted to a wall by the particles adjacent to it, introduced in §8.1
T	shear stress;
	thermodynamic or grain temperature (Chapters 7–10)
T_r	$\equiv I\langle\Omega^2\rangle/(3m_p)$, rotational temperature (Chapter 9)
T_{tot}	$\equiv T + T_r$, total temperature (Chapter 9)
T_w	wall shear stress
T^*	dimensionless grain temperature defined in (8.71)
$\hat{T}$	$\equiv T/v_s^2$, scaled grain temperature
$\hat{U}$	internal energy per unit mass of the material
$\hat{U}'$	true thermal internal energy per unit mass of the material
V_m	material volume
V_D	dimensionless mass flow rate, defined in (3.5)
W	either the half-width or the radius of the bin section of a bunker, depending on whether the cross section is rectangular or circular
$\mathbf{W}$	vorticity tensor or spin tensor, defined by (5.62)
$\mathbf{X}$	position vector of a material point in the reference configuration
α	$\equiv \left(\ell - m\, r^2\right)$, constant introduced in §8.2.1 and §8.3.3
β	filling angle (see Fig. 1.14);
	roughness coefficient, defined in (9.3)

Notation

β_r	angle of repose, the angle formed by the free surface of a heap to the horizontal
γ	orientation of the major principal stress axis (see Fig. 2.11)
γ'	orientation of the major principal stress axis relative to the circumferential direction (see Fig. 3.13)
γ_s	surface tension
$\gamma(\nu)$	$\equiv \Gamma d_\mathrm{p}/(\rho_\mathrm{p} T^{3/2})$, dissipation function (Chapters 8 and 10)
$\gamma_\mathrm{w}(\nu)$	$\equiv \Gamma_\mathrm{w}/(\rho_\mathrm{p} T^{3/2})$, wall dissipation function (Chapters 8 and 10)
δ	angle of wall friction, defined by (2.4); parameter measuring departure from equilibrium, defined in (7.49)
δ_{ij}	Kronecker delta, defined by (A.6)
ϵ	$\equiv 1 - e_\mathrm{p}^2$, inelasticity of particle collisions
ε	$\equiv 1 - \lvert\beta\rvert$, parameter characterizing particle roughness; parameter introduced in (8.75)–(8.78) to elucidate the effect of the walls
ζ	spin viscosity, defined in (9.90); magnitude of $\boldsymbol{\zeta}$ (Chapters 8 and 9)
$\boldsymbol{\zeta}$	$\equiv \hat{\mathbf{g}}/(2\hat{T})^{1/2}$, rescaled velocity of particle 1 relative to particle 2 in Chapters 7 and 9; $\equiv (\mathbf{c} - \mathbf{v}_\mathrm{w})/(2T)^{1/2}$, scaled velocity of a particle relative to the wall in Chapter 8
$\eta(\nu)$	$\equiv \mu/(\rho_\mathrm{p} d_\mathrm{p} \sqrt{T})$, viscosity function (Chapters 8 and 10)
η'	dimensionless vertical coordinate, $= y'/W$ (see Fig. 4.17)
η_1	$\equiv (1 + e_\mathrm{p})/2$ (Chapters 8 and 10)
η_2	$\equiv \frac{1}{2}(1 + \beta)\hat{I}/(1 + \hat{I})$ (Chapter 9)
$\eta_\mathrm{w}(\nu)$	wall viscosity function, defined in $\boldsymbol{\mathcal{S}}_\mathrm{w} = \eta_\mathrm{w} \rho_\mathrm{p} T^{1/2} \mathbf{v}_\mathrm{slip}$ (Chapters 8 and 10)
θ	circumferential coordinate, (see Fig. 3.13); angle of inclination of a chute from the horizontal (§8.3 and 10.2)
θ_0	maximum angle between $\mathbf{k}$ and the inward normal of a wall (see Fig. 8.2)
θ_w	wall angle of the hopper, or of the hopper section (see Figs. 3.13 and 4.1)
$\boldsymbol{\theta}(\psi)$	flux of $\langle\psi\rangle$ due to collisions
ϑ	distribution function of particle spin (Chapter 9)
κ	pseudothermal conductivity
κ_r	conductivity of rotational fluctuational kinetic energy, defined in (9.94)
κ^*	thermal conductivity of a dilute gas of elastic spheres
$\dot{\lambda}$	scalar factor of proportionality, which occurs in the flow rules (2.65) and (5.25)
μ	shear viscosity of a granular material in rapid flow
μ^*	shear viscosity of a dilute gas of elastic spheres
μ_b	bulk viscosity of a granular material in rapid flow
μ_f	shear viscosity of the fluid
μ_r	transport coefficient characterizing the diffusion of intrinsic angular momentum, defined in (9.92)
ν	solids fraction, or volume fraction of solids
ν_d	angle of dilation, defined in (2.83)
ν_lrp	solids fraction corresponding to loose random packing
ν_drp	solids fraction corresponding to dense random packing
ν_s	the scale for the solids fraction, defined below (7.80)

Notation

$\overline{\nu}$	average value of ν across the Couette gap in plane Couette flow
ξ	dimensionless horizontal coordinate (Chapters 2 and 4); dimensionless radial coordinate (Chapter 3); natural strain (Chapter 5); magnitude of $\boldsymbol{\xi}$ (Chapters 7–9);
ξ	$\Omega y/d_{\mathrm{p}}$, scaled distance from lower wall in plane Couette flow §8.2.1; $\Omega(H-y)/d_{\mathrm{p}}$, scaled distance from the free surface in chute flow §8.3.3
ξ_{m}	$\equiv \Omega H^*/2$, value of ξ at the midplane in plane Couette flow §8.2.1
$\boldsymbol{\xi}$	$\equiv \hat{\mathbf{C}}/(2\hat{T})^{1/2}$, rescaled peculiar velocity of particles (Chapters 7–9)
ρ_{p}	particle density, i.e., the density of the solid material forming the particles
ρ_{f}	fluid density
ρ_b	bulk density, $= \rho_{\mathrm{f}}(1-\nu) + \rho_{\mathrm{p}}\,\nu$
ρ	density of the granular material, $= \rho_{\mathrm{p}}\,\nu$
ϱ	$\equiv \mathrm{sgn}(\mathrm{d}v_x/\mathrm{d}y)$ in §10.2
$\boldsymbol{\sigma}$	stress tensor, defined in the compressive sense
$\boldsymbol{\sigma}^{\mathrm{s}}, \boldsymbol{\sigma}^{\mathrm{c}}$	streaming and collisional contributions to the stress tensor in rapid flow
$\boldsymbol{\sigma}^{\mathrm{f}}$	"frictional" stress, defined in §10.1
$\boldsymbol{\sigma}^{\mathrm{k}}$	"kinetic" stress, defined in §10.1
$\hat{\boldsymbol{\sigma}}$	$\equiv \boldsymbol{\sigma}/(\rho_{\mathrm{p}} v_s^2)$, scaled stress tensor (Chapters 7 and 9)
$\sigma_1, \sigma_2, \sigma_3$	principal stresses
σ_i'	effective principal stress, $= \sigma_i - u_w$
σ	mean stress for plane flow, $= (\sigma_1+\sigma_2)/2$
$\overline{\sigma}$	dimensionless normal stress, $= \sigma/(\rho g r_e)$ (Chapter 3); $\sigma/(\rho g W)$ (Chapter 4)
$\mathring{\boldsymbol{\sigma}}$	Jaumann derivative of $\boldsymbol{\sigma}$, defined by (5.64)
τ	deviator stress, $= (\sigma_1-\sigma_2)/2$; relaxation time, defined in (7.26)
τ_c	mean free time, i.e., average time between particle collisions
$\overline{\tau}$	dimensionless deviator stress, $= \tau/(\rho g r_e)$ (Chapter 3); $\tau/(\rho g W)$ (Chapter 4)
$\boldsymbol{\tau}$	external torque per unit mass on the particles (Chapter 9)
ϕ	angle of internal friction, defined by the slope $\sin\phi$ of the critical state line (see (2.47))
ϕ_*	angle of internal friction, defined by (2.15)
ϕ_μ	angle of friction between grains, such that the coefficient of friction is $\tan\phi_\mu$
φ	specularity coefficient, defined in §8.1.1
$\chi(\psi)$	volumetric source of $\langle\psi\rangle$ due to collisions
ψ	any particle property
ψ'	postcollisional value of particle property ψ
$\dot{\psi}_{\mathrm{coll}}$	rate of change of $\langle\psi\rangle$ due to collisions
$\dot{\psi}_{\mathrm{w}}$	rate of transmission of ψ to a wall by particle–wall collisions
$\boldsymbol{\omega}$	angular velocity (Chapter 10); particle "spin," i.e., its angular velocity about its own axes (Chapter 9)
$\overline{\boldsymbol{\omega}}$	mean particle spin
$\boldsymbol{\omega}''$	precollisional spin of a particle in an "inverse collision," as defined in (9.15)
$\hat{\boldsymbol{\omega}}$	$\equiv \boldsymbol{\omega}(I/m_{\mathrm{p}})^{1/2}/v_{\mathrm{s}}$, scaled particle spin
Γ	volumetric dissipation rate of pseudothermal energy due to inelastic particle collisions

$\hat{\Gamma}$ $\equiv \Gamma d_{\mathrm{p}}/(\rho_{\mathrm{p}} v_{\mathrm{s}}^3)$, scaled rate of dissipation of pseudothermal energy
Γ_{w} rate of dissipation of pseudothermal energy due to particle–wall collisions, per unit area of the wall
Γ_r volumetric dissipation rate of rotational fluctuational kinetic energy, defined in (9.30);
volume, due to inelastic particle collisions
Ω constant, $= (\pi/4) - (\phi/2)$;
$\equiv [\pm(L - Mr^2)]^{1/2}$, introduced in §8.2.1
$\boldsymbol{\Omega}$ $\equiv \boldsymbol{\omega} - \overline{\boldsymbol{\omega}}$, peculiar spin of a particle
$\hat{\boldsymbol{\Omega}}$ $\equiv \boldsymbol{\Omega}(I/m_{\mathrm{p}})^{1/2}/v_{\mathrm{s}}$, scaled particle spin
Φ perturbation of the singlet distribution from the local Maxwellian, defined in (7.83)
$\dot{\Phi}$ $= -\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{:}\nabla\mathbf{v}$, stress power

Subscripts
b bin section of a bunker
c critical state
f   fluid property
h hopper section of a bunker
p   particle property
0   value of variable at $O(K^0)$, i.e., O(1) (Chapters 7–9)
K value of variable at $O(K)$ (Chapters 7–9)
ϵ value of variable at $O(\epsilon)$ (Chapters 7 and 9)
ε value of variable at $O(\varepsilon)$ (Chapters 7 and 9)

Superscript
T   transpose