AN INTRODUCTION TO GRANULAR FLOW

The flow of granular materials such as sand, snow, coal, and catalyst particles is a common occurrence in natural and industrial settings. They are important since a large fraction of the materials handled and processed in the chemical, metallurgical, pharmaceutical, and food-processing industries are granular in nature. The mechanics of these materials' flows is not well understood. This book describes the theories for granular flow based mainly on continuum models, although alternative discrete models are also discussed briefly. The level is appropriate for advanced undergraduates or beginning graduate students. The goal is to inform the reader about observed phenomena and some available models and their shortcomings and to visit some issues that remain unresolved. There is a selection of problems at the end of the chapters to encourage exploration, and extensive references are given.

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An Introduction to Granular Flow

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> To our teachers, Professors M. S. Ananth and Roy Jackson

Contents

Preface

page xiii

No	Notation xv	
Т	Introduction	I
	1.1 Examples of Granular Statics and Flow	2
	1.2 Interparticle Forces	11
	1.2.1 Electrostatic and van der Waals Forces	11
	1.2.2 Liquid Bridge or Capillary Forces	14
	1.2.3 Contact Forces	16
	1.2.4 Interparticle Forces in Saturated Granular Materials	19
	1.3 Packing Characteristics	19
	1.3.1 Regular Packings	21
	1.3.2 Random Packings	22
	I.4 Models	22
	1.4.1 Discrete Models	23
	1.4.2 Continuum Models	28
	1.5 Balance Laws for Continuum Models	29
	1.5.1 The Velocity	30
	1.5.2 Integral and Differential Balances	31
	1.5.3 The Mass Balance	32
	1.5.4 The Stress Vector and the Stress Tensor	32
	1.5.5 The Linear Momentum Balance	33
	1.5.6 The Angular Momentum Balance	34
	1.5.7 The Energy Balance	34
	1.6 Statics	36
	1.7 Fluid–Particle Interaction	36
	I.8 Summary	40
2	Theory for Slow Plane Flow	54
	2.1 Qualitative Observations	55
	2.2 The Wall Yield Condition	58
	2.3 The Janssen Solution for the Static Stress Field in a Bin	60
	2.4 The Coulomb Yield Condition	61
	2.5 Generalization of the Coulomb Yield Condition	62
	2.5.1 Invariants and Principal Stresses	63
	2.6 The Mohr–Coulomb Yield Condition	65

Contents

	2.7	The Mohr's Circle for the Two-Dimensional Stress Tensor	65
	2.8	The Relation Between the Coulomb and Mohr–Coulomb	
		Yield Conditions	68
	2.9	Active and Passive States of Stress and the Value of the	
		Janssen K-Factor	69
	2.10	Shear Tests	70
		2.10.1 The Critical State	73
		2.10.2 The Hvorslev Surface	75
		2.10.3 The Roscoe Surface	77
		2.10.4 The Yield Surface	77
	2.11	Yield Surfaces in $\sigma_1 - \sigma_2 - \nu$ Space	79
	2.12	Yield Loci in the $\sigma_1 - \sigma_2$ and $\sigma - \tau$ Planes	81
	2.13	Flow Rules	82
		2.13.1 The Lévy–Mises and the Prandtl–Reuss Equations	84
		2.13.2 The Coaxiality Condition	87
		2.13.3 The Plastic Potential	91
		2.13.4 Positive Dissipation	91
		2.13.5 Associated and Nonassociated Flow Rules	92
	2.14	Equations for Plane Flow	93
		2.14.1 The Mohr's Circle for the Rate of Deformation Tensor	93
		2.14.2 The Coaxiality Condition	94
		2.14.3 The Flow Rule	95
		2.14.4 Implications of the Associated Flow Rule	96
		2.14.5 Rowe's Stress–Dilatancy Relation	96
		2.14.6 Summary of the Governing Equations for Plane Flow	100
	2.15	The Relation Between Yield Loci in the $N-T$ and $\sigma-\tau$ Planes	100
	2.16	The Double-Shearing Model	101
	2.17	Summary	105
3	Flov	v through Hoppers	116
	3.1	Experimental Observations	116
		3.1.1 Flow Rate	116
		3.1.2 Kinematics	121
		3.1.3 Solids Fraction Profiles	122
	3.2	Theory for Steady, Plane Flow	123
		3.2.1 The Critical State Approximation	123
	3.3	The Smooth Wall, Radial Gravity (SWRG) Problem	125
	3.4	The Effect of Wall Roughness	131
	3.5	Solutions with Allowance for Rough Walls and Vertical Gravity	134
		3.5.1 The Brennen–Pearce Solution	134
		3.5.2 The Radial Stress and Velocity Fields	138
		3.5.3 Linearized Stability Analysis	141
		3.5.4 Downward Integration from the Radial Fields	143
		3.5.5 The Successive Approximation Procedure	145
	3.6	A Re-Examination of the Exit Condition	149
	3.7	An Alternative Exit Condition	152
	3.8	The Smooth Wall, Radial Gravity Problem for Compressible Flow	154
	3.9	Summary	160
4	Flov	v through Wedge-Shaped Bunkers	166
	4.1	Experimental Observations	166

viii

Contents

	4	4.1.1	Flow Regimes	166
	4	4.1.2	Kinematics	166
	4	4.1.3	Wall Stresses	175
	4	4.1.4	Bins	178
	4.2 I	Models	s for Bunker Flow	182
	4	4.2.1	The Bin Section	183
	4	4.2.2	The Transition Region	190
	4	4.2.3	The Hopper Section	203
	4.3 \$	Summa	ary	208
5	The	ory fo	or Slow Three-Dimensional Flow	213
	5.1 (Consti	tutive Equations Involving a Yield Condition	213
	5	5.1.1	The Yield Condition	213
	5	5.1.2	Symmetry Considerations	220
	5	5.1.3	Conventional Triaxial Tests	220
	5	5.1.4	Isotropic Compression Tests	222
	5	5.1.5	Compression Tests	223
	ļ	5.1.6	Cubical Triaxial Tests	225
	ļ	5.1.7	Comparison of Yield Conditions with Data	226
	ļ	5.1.8	Flow Rules	226
	5	5.1.9	Data Related to Flow Rules	228
	5	5.1.10	Steady, Fully Developed Flow of a Rigid-Plastic Material	229
	5	5.1.11	One-Dimensional Deformation of a Rigid-Plastic Material	230
	5.2 (Consti	tutive Equations That Do Not Involve a Yield Condition	233
		5.2.1	Hypoelastic and Hypoplastic Models	233
		5.2.2	Some Features of (5.66)	235
		5.2.3	Steady, Fully Developed Flow of a Hypoelastic Material	236
		5.2.4	One-Dimensional Deformation of a Hypoelastic Material	236
	5.3 9	Summa	ary	239
6	Flov	v thro	ugh Axisymmetric Hoppers and Bunkers	249
	6.I I	Experi	mental Observations	249
	6	5.1.1	Flow Rate	249
	6	6.1.2	Velocity Profiles	249
	6	6.1.3	Density Profiles	25 I
	6	6.1.4	Flow Patterns	253
	6	6.1.5	Stress Profiles	256
	6.2	Theory	y for Steady, Axisymmetric Flow Through a Hopper	259
	6	5.2.I	The Haar–von Karman Hypothesis	261
	6	5.2.2	The Radial Stress and Velocity Fields for the	
			Mohr–Coulomb Yield Condition and the Haar–von	
			Karman Hypothesis	264
	6	6.2.3	The Drucker–Prager Yield Condition and Levy's Flow Rule	266
	6	5.2.4	Comparison of Predicted and Measured Velocity Profiles	269
	(6.2.5	Criteria for Mass Flow	269
	6.3	А НуЬ	rid Hypoplastic-Viscous Model	273
	6.4	The Ki	nematic Model for Batch Discharge from a Bin	276
	6.5 \$	Summa	ary	280
7	The	ory fo	or Rapid Flow of Smooth, Inelastic Particles	285
	7.I I	Prelimi	inaries and Scaling	285

Contents

		7.1.1	Model for Inelastic Collisions	288
		7.1.2	Hydrodynamic Description of Rapid Granular Flows	289
	7.2	Heuris	tic Hydrodynamic Theory for High-Density Flows	290
		7.2.1	Application to Uniform Plane Shear	292
	7.3	Kinetio	c Theory for a Granular Gas of Smooth	
		Inelast	ic Particles	295
		7.3.I	Statistical Preliminaries	295
		7.3.2	The Evolution of $f^{(1)}$	297
		7.3.3	The Equilibrium Distribution Function	300
		7.3.4	The Departure from Equilibrium	301
		7.3.5	Maxwell Transport Equation	302
		7.3.6	The Equations of Motion	305
		7.3.7	The Chapman–Enskog Expansion	307
		7.3.8	Constitutive Relations at Leading Order	310
		7.3.9	Distribution Function at $O(K)$	311
		7.3.10	Solution for Φ_{K}	316
		7.3.11	Constitutive Relations at $O(K)$	319
		7.3.12	Distribution Function and Constitutive Relations at O(ϵ)	321
		7.3.13	Constitutive Relations to First Order in K and ϵ	323
	7.4	Anisot	ropy of the Microstructure	325
	7.5	Extens	sion to Granular Mixtures	326
	7.6	Summa	ary and Discussion	328
8	An	alysis o	of Rapid Flow in Simple Geometries	331
	8. I	Bound	ary Conditions at Solid Walls	33 I
		8.1.1	Heuristic Theory	332
		8.1.2	Kinetic Theory	334
	8.2	Plane (Couette Flow	339
		8.2.I	Predictions of the High-Density Theory	340
		8.2.2	Some Features of the High-Density Solutions	345
		8.2.3	Predictions of the Kinetic Theory	346
	8.3	Flow i	n Inclined Chutes	349
		8.3.1	Some Experimental Observations of Chute Flow	351
		8.3.2	Analysis of Steady, Fully Developed Flow	355
		8.3.3	High-Density Theory	356
		8.3.4	Some Features of the High-Density Solutions	358
		8.3.5	Predictions of the Kinetic Theory	359
	8.4	Stabilit	y of Rapid Shear Flows	366
		8.4.1	Stability of Unbounded Plane Shear Flow	366
		8.4.2	Stability of Plane Couette Flow	369
	8.5	Summ	ary	371
9	Th	eory fo	or Rapid Flow of Rough,	
	Ine	lastic	Particles	374
	9.I	Collisi	on Models for Rough Particles	375
	9.2	Equation	ons of Motion for a Granular Gas of Rough, Inelastic Spheres	378
	9.3	The Ve	elocity Distribution Function	382
		9.3.1	Nearly Elastic, Nearly Perfectly Rough Particles	382
		9.3.2	Nearly Elastic, Nearly Smooth Particles	386
	9.4	Const	itutive Relations up to First Order in K , ϵ , and ε	389
		9.4.1	Nearly Elastic, Nearly Perfectly Rough Particles	389

х

Contents

	9.4.2 Nearly Elastic, Nearly Smooth Particles	390
	9.5 Summary	392
10	Hybrid Theories	394
	10.1 The Frictional-Kinetic Model	395
	10.2 Application to Flow in Chutes	396
	10.3 Other Hybrid Models	400
	10.4 Summary	401
Δn	pendix A: Operations with Vectors and Tensors	403
- P	A.I. Vectors	403
	A.2 The Summation Convention	404
	A.3 The Scalar Product of Two Vectors	404
	A.4 Second-Order Tensors	405
	A.4.1 The Unit Tensor	406
	A.4.2 The Trace of a Second-Order Tensor	406
	A.5 Cartesian Tensor Notation	407
	A.6 Third and Higher Order Tensors	407
	A.6.1 The Alternating Tensor	407
	A.7 Operations with Vectors and Tensors	407
	A.7.1 The Vector Product of Two Vectors	407
	A.7.2 The Product of Two Second-Order Tensors	408
	A.7.3 The Scalar Product of Two Second-Order Tensors	408
	A.7.4 The Transpose of a Tensor	409
	A.7.5 The Inverse of a Second-Order Tensor	409
	A.7.6 The Determinant of a Second-Order Tensor	409
	A.7.7 Orthogonal Second-Order Tensors	410
	A.7.8 The Gradient Operator	411
	A.7.9 The Gradient of Scalars and Vectors	411
	A.7.10 The Divergence of a Second-Order Tensor	412
	A.7.11 The Curl of a Vector	412
	A.8 Equations in Orthogonal Curvilinear Coordinate Systems	412
	A.8.1 Cylindrical Coordinates	412
	A.8.2 Spherical Coordinates	414
Ар	pendix B: The Stress Tensor	418
۸n	nendix C: Hyperbolic Partial Differential Equations of	
Αþ	Eirst Order	420
	C L Solution by the Method of Characteristics	474
	e.r. solution by the richied of characteristics	
Ар	pendix D: Jump Balances	428
	D.I The Jump Mass Balance	428
_		
Ар	pendix E: Discontinuous Solutions of Hyperbolic Equations	430
	E.I Weak Solution	430
	E.2 Jump Conditions	431
	E.3 Jump Conditions for Linear Equations	432
Ар	pendix F: Proof of the Coaxiality Condition	434
An	pendix G: Material Frame Indifference	439
· •P	G.I Change of Frame	439

Contents

G.2 Frame Indifferent Scalars, Vectors, and Tensors	441	
G.2.1 Scalars	441	
G.2.2 Vectors	441	
G.2.3 Second-Order Tensors	442	
G.3 The Principle of Material Frame Indifference	443	
G.4 An Alternative Interpretation of a Change of Frame	443	
Appendix H: The Evaluation of Some Integrals	449	
H.I Integration Over k	449	
H.2 Integration Over k for Boundary Conditions	450	
H.3 Change of Variables	45 I	
H.4 Volume Integrals	452	
H.5 Gaussian Integrals	453	
Appendix I: A Brief Introduction to Linear Stability Theory	454	
Appendix J: Pseudo Scalars, Vectors, and Tensors	456	
Appendix K: Answers to Selected Problems	459	
References	463	
Index 4		

Preface

The flow of granular materials such as sand, snow, coal, and catalyst particles is a common occurrence in natural and industrial settings. Unfortunately, the mechanics of these materials is not well understood. Experiments reveal complex and, at times, unexpected behavior, whereas existing theories are often tentative and do not represent the entire range of observed behavior. Nevertheless, significant advances have been made in the understanding of the mechanics of granular flows, and the time is ripe for an account of experimental observations and theoretical models pertaining to flow in relatively simple geometries.

The importance of understanding granular flows need not be overstated – a large fraction of the materials handled and processed in the chemical, metallurgical, pharmaceutical, and food-processing industries are granular in nature. The flow and transportation of these materials are often critical operations in these processes. In most cases, the design of processes and equipment is based largely on experience and empirical rules. An appreciation of the underlying principles may be helpful in developing better design and operating procedures.

Some of the early investigations of granular flow were motivated by the need to understand the deformation of soils subjected to external loads, such as large structures. The deformation rates in these processes are usually very small. Theoretical models for these slow flows have increased in sophistication and complexity over the years, borrowing concepts from metal plasticity and soil mechanics. A contrasting picture of granular flow has emerged over the last three decades. This is believed to be applicable to rapid flows, where the deformation rates are large. Models for rapid flows have been based mainly on the kinetic theory of dense gases, with suitable modifications to account for the inelasticity of interparticle collisions and particle roughness. These models assume that momentum transfer occurs by collisions of short duration between particles and by free flight of particles between collisions. In contrast, momentum transfer during slow flow occurs via contacts of a longer duration between particles that slide and roll relative to each other. A realistic picture of particle interactions that encompasses these extremes is not available at present, barring some tentative attempts.

The theories described here for granular flow are based mainly on continuum models, although alternative discrete models are also discussed very briefly. Chapter 1 describes the qualitative behavior of granular materials in various situations and formulates balance laws. Chapter 2 discusses constitutive equations for slow, plane flow. Chapters 3 and 4 deal with flow in wedge-shaped hoppers and bunkers, respectively; after summarizing some of the experimental observations, the equations formulated in Chapter 2 are used to construct approximate solutions. Chapter 5 deals with constitutive equations for slow three-dimensional flow; these are applied to flow through axisymmetric hoppers and bunkers in Chapter 6. Using kinetic theory, constitutive equations for rapid granular flow are formulated

xiii

Preface

in Chapter 7 and applied to flow between parallel plates and down inclined chutes in Chapter 8. The kinetic theory is extended to accommodate particle roughness in Chapter 9. Finally, a few tentative attempts to model flow in the intermediate regime, where a range of interparticle contact times is present, are desribed in Chapter 10. For the reader interested in details, extensive references are given. Some problems are included at the end of most of the chapters, to enable the reader to either fill in details that are omitted in the book or extend the material.

Overall, we wish to inform the reader about observed phenomena, some of the available models and their shortcomings, and unresolved issues in granular flow. We describe in detail only a very small number of "animals" in the vast and rapidly expanding granular zoo, but we hope that the material presented here will stimulate readers to learn more about this field. Many interesting phenomena and puzzles lurk in topics such as vibrated beds, segregation in rotating cylinders, landslides and sand dunes, the behavior of cohesive powders, and fluid–particle flows, which either have not been discussed here or are mentioned only in passing.

This book deals with granular flow at a level that should be suitable for senior undergraduate and postgraduate students. A knowledge of undergraduate-level fluid and solid mechanics is desirable but not essential. As the book provides a reasonably self-contained introduction to the subject, it may also be useful to new entrants to the field. It is hoped that a study of the simple problems discussed here may prepare the reader for a better understanding of more realistic and complex situations, which are often encountered in industries. Parts of this book are based on graduate courses taught at Princeton University, Indian Institute of Science, and California Institute of Technology.

We request readers to inform us by email, (kesava@chemeng.iisc.ernet.in or prnott@chemeng.iisc.ernet.in) if they come across errors. We shall also be grateful for suggestions that may improve the book. An errata for the book and supplementary information, such as solutions to problems (for instructors) and additional problems, may be found at the catalogue page of the book (http://www.cambridge.org/us/catalogue/ catalogue.asp?isbn=0521571669).

We are grateful to our teacher Prof. R. Jackson for introducing us to this field and to Dr. R. M. Nedderman for prodding one of us to write a book on it. Prof. S. Sundaresan had originally planned to be one of the coauthors. He opted out later, but we are very grateful to him for his comments and suggestions on some of the chapters. We are grateful to professors B. Ananthanarayan, I. Goldhirsch, R. Jackson, C. S. Jog, H. S. Mani, S. Ramaswamy, and D. Sen for helpful discussions. We express our heartfelt thanks to our editors, Ms. Florence Padgett, Mr. Roger Astley, and Mr. Peter Gordon, who showed extraordinary patience in dealing with our repeated requests for extensions. We are grateful to Ms. B. G. Girija, Mr. K. Venugopal, Mr. Gautam Parthasarathy, and Mr. Vishwajeet Mehandia for drawing some of the figures; Mr. P. T. Raghuram, Ms. Shruti Seshadri, Mr. Alok Srivastava, Dr. S. Venugopal, and Prof. Sanjeev Gupta for assistance with photographs; the staff of Aptara for promptly responding to our queries regarding LaTeX, and Mr. Anoop Chaturvedi of Aptara for responding to many queries during our review of the copyedited manuscript. Special thanks are due to various publishers and societies who have permitted us to reproduce figures from their journals, to professors A. Drescher and R. P. Behringer for permitting us to use some of their photographs, and to Prof. D. W. Agar for translating the titles of some articles from French and German to English. Finally, we wish to record our gratitude to and deep appreciation for our families, who tolerated our preoccupation with this work for an extended period of time.

Notation

$a_1 - a_6$	constants of $O(1)$ in the heuristic high-density theory for rapid flows, §7.2 and 8.1.1
a_n, b_n	coefficients in the expansion of \mathcal{A}, \mathcal{B} in (7.126)
b	body force per unit mass
_b	scaled body force per unit mass, defined in (7.79)
b_{*0}	$\equiv b_*(\theta = 0)$, constant occurring in the boundary conditions for the radial
	stress field (see (6.32) and (6.48))
С	cohesion
c_w	adhesion
c_1, c_2, c_3	principal compressive rates of deformation, i.e., the eigenvalues of C
c_*	$=(c_1+c_2)/2$
c	translation velocity of particles
ĉ	scaled translation velocity, defined in (7.79)
c ′	postcollisional translation velocity of a particle
c ″	precollisional translation velocity of a particle in an "inverse collision," as
	defined in §7.3.2
$d_{\rm p}$	particle diameter
$d_{ m w}$	diameter of a wall hemisphere (see Fig. 8.2)
\overline{d}	$\equiv \frac{1}{2}(d_{\rm p} + d_{\rm w})$
det	determinant of a tensor or a matrix, defined in (A.37) and (A.41)
ep	coefficient of restitution for particle collisions, defined in (7.3)
$e_{\rm w}$	coefficient of restitution for particle-wall collisions
\mathbf{e}_i	basis vector for a Cartesian coordinate system
$f, f^{(1)}$	single-particle probability distribution function of particle position and velocity, sometimes referred to as the singlet distribution
f^0	Maxwell–Boltzmann distribution based on the local density, velocity, and temperature
\hat{f}	scaled singlet distribution, defined in (7.79)
f'	singlet distribution for the postcollision velocity of a "direct collision"
	(see §7.3.2)
$f''_{c(2)}$	singlet distribution for the precollision velocity of an "inverse collision"
$f^{(2)}$	probability distribution function of the position and velocity of a pair of particles
\dot{f}_{coll}	rate of change of $f^{(1)}$ due to collisions
f_d	drag force exerted by the fluid on a single particle
g	acceleration due to gravity;
	$\equiv n^{(2)}/n^2$, pair distribution function

Notation

g	translation velocity of particle 1 relative to particle 2
ĝ	\equiv g/v _s , scaled relative translation velocity
g′	postcollisional value of g
$g_0(\theta)$	function defining the radial stress field
$g_0(v)$	equilibrium pair distribution function at contact (Chapters 7–10)
$g_{\rm w}(v)$	enhancement in the number density of particles in contact with the wall,
8	defined in §8.1.2
h	relative velocity of the point of contact of particle 1 with respect to that of
	particle 2, given by (9.1)
k_x, k_y, k_z	wavenumbers for the x, y, and z directions ($\S8.4$)
k	unit vector along the line joining the center of particle 1 to the center of
	particle 2 (see Fig. 7.3);
	unit vector along the line joining the center of a wall hemisphere to the center
	of a colliding particle (see Fig. 8.2)
k ″	k for an "inverse collision" (see Fig. 7.6)
l	$\equiv (1 - e_w^2) a_6/a_3$, constant introduced in §8.2.1 and §8.3.3
т	$\equiv a_1^2/(a_3 a_5 \varphi)$, constant introduced in §8.2.1 and §8.3.3
m_i	mass of particle <i>i</i>
m _p	mass of particle
<i>m</i>	mass flow rate per unit width in chute flow (§8.3 and 10.2)
m̀*	dimensionless mass flow rate, defined in (8.81)
m_{ℓ}	loading, that is, mass of material per unit area of the chute base ($\S8.3.1$ and 10.2)
m*	dimensionless loading defined in (8.81)
n	number density of narticles
$n^{(2)}$	probability distribution function of the positions of a pair of particles
n	unit normal to a bounding surface
n	mean stress defined in (5.6): pressure in rapid granular flow
P n	mean stress at a critical state
p_c	fluid pressure
$P_{\rm f}$	invariant of the stress tensor defined in (5.6)
9 0	nseudothermal energy flux:
Ч	energy flux (Chapter 1)
a^{s} , a^{c}	streaming and collisional contributions to the pseudothermal energy flux
â	$\equiv \mathbf{q}/(\rho v^3)$, scaled pseudothermal energy flux (Chapter 7)
ч 0	nseudothermal energy flux: referred to as a after (7.11)
Apt a	flux of rotational fluctuational kinetic energy defined in (9.20)
Υr α	flux of fluctuational kinetic energy to the wall due to particle-wall collisions
\mathbf{q}_{W}	flux of true thermal energy
ч r	radial coordinate measured from the virtual anex of the honner (see Fig. 3.13):
1	$\equiv N/S$, stress ratio introduced in §8.2.1
r _e	radial coordinate corresponding to the edge of the exit slot
r	position vector of particle
ŕ	scaled position vector, defined (7.79)
S	$= (c_1 - c_2)/2$, the magnitude of the maximum shear rate in plane flow;
	mean free path of particles (Chapters 7 and 8)
S i	dimensionless arc length measured along the <i>i</i> th characteristic
\overline{S}_{W}	mean spacing between wall hemispheres (see Fig. 8.2)
Sii	deviatoric stress component, $= \sigma_{ii} - p\delta_{ii}$
s.	$\equiv \hat{\mathbf{\Omega}}/(2\hat{T})^{1/2}$, rescaled peculiar spin (Chapter 9)
s'	deviatoric stress tensor. = $\sigma - p \mathbf{I}$
-	r

Notation

î	scaled time, defined in (7.79)
t _(n)	stress vector, = force per unit area exerted on a surface with unit inward
< /	normal n
tr	trace of a tensor, defined by (A.18)
u_w	pore pressure or pressure in excess of the atmospheric pressure which is exerted by the fluid on the particles
v_s	velocity scale, used to scale variables in (7.79)
v^*	dimensionless velocity, defined in (8.71) for plane Couette flow and (8.104)
	for chute flow
V	velocity of a material point, or mean velocity $\langle c \rangle$ of a collection of particles
\mathbf{v}_i	velocity of particle i
V	$\equiv \mathbf{v}/v_{\rm s}$, scaled mean velocity
\mathbf{V}_{W}	velocity of a wall
V _{slip}	$\equiv \mathbf{v} - \mathbf{v}_{w}$, velocity slip at a wall
w	root-mean-square fluctuation velocity of particle (Chapters 7 and 8)
W	local averaged velocity of the fluid phase relative to the particle phase
<i>y</i> *	dimensionless distance, defined in (8.71) for plane Couette flow and (8.104) for chute flow
A	dimensionless cross-sectional area, scaled by W^2
$\mathcal{A}, \mathcal{B}, \mathcal{R}$	functions of ξ in the expression for Φ_K in (7.125) and (9.70)
C	cohesion (Section 2.2); rate of compression in the direction of the normal to a
	plane, defined in (2.74)
C_{ii}	component of the rate of deformation tensor C
Ċ	rate of deformation tensor, defined in the compressive sense by (2.49);
	\equiv c – v . peculiar velocity of a particle
Ĉ	$= \mathbf{C}/v_{\rm s}$ scaled neculiar velocity of particles
D/Dt	$= c_{1} v_{s}$, section period version version particles material derivative defined by (1.23)
D/Di	either the diameter or the width of the exit slot of a hopper or hunker
D	(see Fig. 3.1)
n	$-\mathbf{C}$ rate of deformation tensor defined by (1.50)
n n	$= \mathbf{D} H/v_{\rm s}$ scaled rate of deformation tensor
Б	$=$ D Π/v_s , scaled fate of deformation tensor drag force per unit volume of the sugrangian evented by the fluid phase on the
F _d	particle phase
Ĝ	$\equiv (\hat{\mathbf{C}}_1 + \hat{\mathbf{C}})/2$, scaled mean peculiar velocity of colliding particles
Н	head or height of the granular material above the exit slot of a hopper or
	bunker (see Fig. 3.2);
	characteristic macroscopic length scale in the Chapman–Enskog expansion (Chapters 7 and 9);
	distance between plates in plane Couette flow;
	thickness of flowing layer in chute flow
H^*	$\equiv H/d_{\rm p}$, dimensionless Couette gap in plane Couette flow. §8.2.1:
	dimensionless flow depth in chute flow. 88.3
$H_{h,d}^*$	value of H^* for chute flow predicted by the high-density theory at a given
11 nd	angle of inclination θ
Ι	moment of inertia of a particle (Chapter 9)
I_1, I_2, I_3	principal invariants of the stress tensor, defined in (5.2)
Î	$\equiv 4I/(m_{\rm p}d_{\rm p}^2)$, dimensionless moment of inertia of a particle (Chapter 9)
Ι	unit (identity) tensor
J_2	second invariant of the deviatoric stress tensor, defined by (5.9)
$\overline{\mathcal{J}}$	Jacobian for the transformation of one set of variables to another
-	

xvii

Notation

J	impulse per unit mass on particle 2 during collision with particle 1
Κ	coefficient of earth pressure at rest, i.e., the ratio of the horizontal normal
	stress to the vertical normal stress;
	$\equiv Kn^{-1}$, inverse Knudsen number (Chapters 7–10)
$\mathcal{K}(v)$	$\equiv \kappa / (\rho_n d_n \sqrt{T})$, thermal conductivity function (Chapters 8 and 10)
Kn	$\equiv H/s$, Knudsen number, ratio of macroscopic to microscopic length scales
М	mass flow rate
L	$\equiv a_4(1-e_\pi^2)/a_3$, constant introduced in §8.2.1 and §8.3.3
$L^{(p)}(\mathbf{r})$	Generalized Laguerre polynomials or Sonine polynomials used in 87.3.10
$\mathcal{L}_n(\mathcal{X})$	linearized Boltzmann operator, defined in (7.113)
<i>L</i> r	linearized Boltzmann operator for rough particles, defined in (9.47)
\mathcal{L}^*	linearized operator defined in (9.74)
М	$\equiv a_1^2/(a_2 a_3)$, constant introduced in §8.2.1 and §8.3.3
Μ	couple stress, defined in (9.27)
Ν	normal stress
N_b	normal stress exerted on the bin wall
N_h	normal stress exerted on the hopper wall
N_w	normal stress exerted on the wall of a bin or hopper
\mathcal{N}	normal stress on the walls in plane Couette flow (see Fig. 8.4)
Р	dimensionless perimeter of the bin section, scaled by W
\mathcal{P}	$\equiv p/(\rho_{\rm p}T)$, pressure function
Q	orthogonal tensor, defined in (5.60)
R	particle radius
Re	Reynolds number
S	rate of shear in the direction of a unit vector t which is tangential to a surface, defined in (2.74)
S_m	bounding surface of a material volume
S	shear stress on the walls in plane Couette flow (see Fig. 8.4)
\mathcal{S}_{w}	stress transmitted to a wall by the particles adjacent to it, introduced in §8.1
Т	shear stress;
	thermodynamic or grain temperature (Chapters 7-10)
T_r	$\equiv I \langle \Omega^2 \rangle / (3m_p)$, rotational temperature (Chapter 9)
$T_{\rm tot}$	$\equiv T + T_{\rm r}$, total temperature (Chapter 9)
T_w	wall shear stress
T^*	dimensionless grain temperature defined in (8.71)
T	$\equiv T/v_s^2$, scaled grain temperature
U	internal energy per unit mass of the material
\widehat{U}'	true thermal internal energy per unit mass of the material
V_m	material volume
V_D	dimensionless mass flow rate, defined in (3.5)
W	either the half-width or the radius of the bin section of a
	bunker, depending on whether the cross section is
	rectangular or circular
W	vorticity tensor or spin tensor, defined by (5.62)
X	position vector of a material point in the reference configuration
α	$\equiv (\ell - m r^2)$, constant introduced in §8.2.1 and §8.3.3
β	filling angle (see Fig. 1.14);
	roughness coefficient, defined in (9.3)

xviii

Notation

β_r	angle of repose, the angle formed by the free surface
	of a heap to the horizontal
γ	orientation of the major principal stress axis (see Fig. 2.11)
γ'	orientation of the major principal stress axis relative to the
	circumferential direction (see Fig. 3.13)
γ_s	surface tension
$\gamma(\nu)$	$\equiv \Gamma d_{\rm p}/(\rho_{\rm n} T^{3/2})$, dissipation function (Chapters 8 and 10)
$\gamma_{\rm w}(\nu)$	$\equiv \Gamma_{\rm w}/(\rho_{\rm p} T^{3/2})$, wall dissipation function (Chapters 8 and 10)
δ	angle of wall friction, defined by (2.4);
	parameter measuring departure from equilibrium, defined in (7.49)
δ_{ii}	Kronecker delta, defined by (A.6)
ϵ	$\equiv 1 - e_{\rm p}^2$, inelasticity of particle collisions
ε	$\equiv 1 - \beta $, parameter characterizing particle roughness;
	parameter introduced in (8.75) – (8.78) to elucidate the effect of the walls
ζ	spin viscosity, defined in (9.90);
	magnitude of ζ (Chapters 8 and 9)
ζ	$\equiv \hat{\mathbf{g}}/(2\hat{T})^{1/2}$, rescaled velocity of particle 1 relative to particle 2 in Chapters 7
-	and 9;
	$\equiv (\mathbf{c} - \mathbf{v}_{w})/(2T)^{1/2}$, scaled velocity of a particle relative to the wall in
	Chapter 8
$\eta(v)$	$\equiv \mu/(\rho_{\rm p} d_{\rm p} \sqrt{T})$, viscosity function (Chapters 8 and 10)
η'	dimensionless vertical coordinate, $= y'/W$ (see Fig. 4.17)
η_1	$\equiv (1 + e_p)/2$ (Chapters 8 and 10)
η_2	$\equiv \frac{1}{2}(1+\beta)\hat{I}/(1+\hat{I}) \text{ (Chapter 9)}$
$\eta_{\rm w}(\nu)$	wall viscosity function, defined in $S_{\rm w} = \eta_{\rm w} \rho_{\rm p} T^{1/2} \mathbf{v}_{\rm slip}$ (Chapters 8 and 10)
θ	circumferential coordinate, (see Fig. 3.13);
	angle of inclination of a chute from the horizontal (§8.3 and 10.2)
$ heta_0$	maximum angle between k and the inward normal of a wall (see Fig. 8.2)
θ_w	wall angle of the hopper, or of the hopper section
	(see Figs. 3.13 and 4.1)
$oldsymbol{ heta}(\psi)$	flux of $\langle \psi \rangle$ due to collisions
θ	distribution function of particle spin (Chapter 9)
К	pseudothermal conductivity
κ _r	conductivity of rotational fluctuational kinetic energy, defined in (9.94)
κ* ·	thermal conductivity of a dilute gas of elastic spheres
λ	scalar factor of proportionality, which occurs in the flow rules (2.65) and (5.25)
μ	shear viscosity of a granular material in rapid flow
μ^*	shear viscosity of a dilute gas of elastic spheres
$\mu_{\rm b}$	bulk viscosity of a granular material in rapid flow
μ_{c}	shear viscosity of the fluid
μ_r	transport coefficient characterizing the diffusion of intrinsic angular
	momentum, defined in (9.92)
ν	solids fraction, or volume fraction of solids
v_d	angle of dilation, defined in (2.83)
v_{lrp}	solids fraction corresponding to loose random packing
v_{drp}	solids fraction corresponding to dense random packing
v_s	the scale for the solids fraction, defined below (7.80)

xix

Notation

$\overline{\nu}$	average value of ν across the Couette gap in plane Couette flow
ξ	dimensionless horizontal coordinate (Chapters 2 and 4);
	dimensionless radial coordinate (Chapter 3);
	natural strain (Chapter 5);
	magnitude of $\boldsymbol{\xi}$ (Chapters 7–9);
ξ	$\Omega y/d_p$, scaled distance from lower wall in plane Couette flow §8.2.1; $\Omega(H - y)/d_p$, scaled distance from the free surface in chute flow §8.3.3
ξm	$\equiv \Omega H^*/2$, value of ξ at the midplane in plane Couette flow §8.2.1
ξ	$\equiv \hat{\mathbf{C}}/(2\hat{T})^{1/2}$, rescaled peculiar velocity of particles (Chapters 7–9)
$\rho_{\rm p}$	particle density, i.e., the density of the solid material forming the particles
$\rho_{\rm f}$	fluid density
ρ_b	bulk density, = $\rho_{\rm f}(1-\nu) + \rho_{\rm p} \nu$
ρ	density of the granular material, $= \rho_n v$
Q	$\equiv \operatorname{sgn}(dv_x/dy) \text{ in } \S10.2$
σ	stress tensor, defined in the compressive sense
σ^{s}, σ^{c}	streaming and collisional contributions to the stress tensor in rapid flow
σ^{f}	"frictional" stress, defined in §10.1
σ^{k}	"kinetic" stress, defined in §10.1
σ	$\equiv \sigma / (\rho_p v_s^2)$, scaled stress tensor (Chapters 7 and 9)
$\sigma_1, \sigma_2, \sigma_3$	principal stresses
σ'_i	effective principal stress, $= \sigma_i - u_w$
σ	mean stress for plane flow, = $(\sigma_1 + \sigma_2)/2$
$\overline{\sigma}$	dimensionless normal stress, $= \sigma/(\rho g r_e)$ (Chapter 3);
0	$\sigma/(\rho g W)$ (Chapter 4)
σ	Jaumann derivative of σ , defined by (5.64)
τ	deviator stress, = $(\sigma_1 - \sigma_2)/2$;
_	relaxation time, defined in (7.26)
τ_c	mean free time, i.e., average time between particle confisions dimensionlass deviator strass $-\pi/(\cos t)$ (Chapter 2):
ι	unitensionless deviator stress, $= t/(\rho g r_e)$ (Chapter 5), $\tau/(\rho q W)$ (Chapter 4)
τ	external torque per unit mass on the particles (Chapter 9)
с Ф	angle of internal friction defined by the slope $\sin \phi$ of
Ψ	the critical state line (see (2.47))
ϕ_*	angle of internal friction, defined by (2.15)
ϕ_{μ}	angle of friction between grains, such that the coefficient of
	friction is $\tan \phi_{\mu}$
φ	specularity coefficient, defined in §8.1.1
$\chi(\psi)$	volumetric source of $\langle \psi \rangle$ due to collisions
ψ	any particle property
ψ'	postcollisional value of particle property ψ
$\dot{\psi}_{\rm coll}$	rate of change of $\langle \psi \rangle$ due to collisions
$\psi_{ m w}$	rate of transmission of ψ to a wall by particle–wall collisions
ω	angular velocity (Chapter 10);
	particle "spin," i.e., its angular velocity about its own axes (Chapter 9)
$\overline{\omega}_{\mu}$	mean particle spin
ω″	precollisional spin of a particle in an "inverse collision," as defined in (9.15)
ω	$\equiv \omega (1/m_{\rm p})^{1/2}/v_{\rm s}, \text{ scaled particle spin}$
1	volumetric dissipation rate of pseudothermal energy due to inelastic particle collisions

Notation

- $\hat{\Gamma} \equiv \Gamma d_{\rm p}/(\rho_{\rm p} v_{\rm s}^3)$, scaled rate of dissipation of pseudothermal energy
- Γ_{w} rate of dissipation of pseudothermal energy due to particle–wall collisions, per unit area of the wall
- Γ_r volumetric dissipation rate of rotational fluctuational kinetic energy, defined in (9.30);
 - volume, due to inelastic particle collisions
- Ω constant, = $(\pi/4) (\phi/2)$;
- $\equiv [\pm (L Mr^2)]^{1/2}$, introduced in §8.2.1
- $\Omega \equiv \omega \overline{\omega}$, peculiar spin of a particle
- $\hat{\mathbf{\Omega}} \equiv \mathbf{\Omega}(I/m_{\rm p})^{1/2}/v_{\rm s}$, scaled particle spin
- Φ perturbation of the singlet distribution from the local Maxwellian, defined in (7.83)
- $\dot{\Phi} = -\boldsymbol{\sigma}^{\mathrm{T}}: \nabla \mathbf{v}$, stress power

Subscripts

- b bin section of a bunker
- *c* critical state
- f fluid property
- *h* hopper section of a bunker
- p particle property
- 0 value of variable at $O(K^0)$, i.e., O(1) (Chapters 7–9)
- K value of variable at O(K) (Chapters 7–9)
- ϵ value of variable at O(ϵ) (Chapters 7 and 9)
- ε value of variable at O(ε) (Chapters 7 and 9)

Superscript

T transpose