1

Fundamentals of magnetism

1.1 Introduction

Earlier texts on permanent magnets have opened with historical reviews of these materials (Hadfield, 1962; McCaig, 1977; Parker, 1990; Parker and Studders, 1962). In this book the design of modern permanent magnets is emphasized, and so initially the development of the relationships that are required to model today's materials for a variety of common applications is considered. To the extent that a historical review is provided in this chapter, it is of those fundamental equations of electromagnetism that are needed to understand the performance of magnets in circuits and devices.

There are many properties of a permanent magnet that are considered in its design for a magnetic device, but most often it is the *demagnetization curve* that initially determines its suitability for the task. Its shape contains information on how the magnet will behave under static and dynamic operating conditions, and in this sense the material characteristic will constrain what can be achieved in the device design.

The *B versus H* loop of any permanent magnet has some portions which are almost linear, and others that are highly non-linear. The shapes of these *B versus H* loops, or at least the *demagnetization* portions of them, tell the designer a lot about the suitability of the material for a given application. A brief derivation of the *B versus H* loop is presented, to illustrate the microscopic mechanisms that determine the macroscopic performance of a magnet.

The fundamental theory describing permanent magnets is similar to that for soft magnetic materials. A magnet is permanent (sometimes called *hard*) if it will alone support a useful flux in the air gap of a device, whereas it is *soft* if it can only do so with the aid of an external electrical or magnetic input. The most basic parameter of either type of material is the *magnetic dipole moment*.

1.2 Magnetic dipole moment

The magnetic dipole moment may be modeled in a similar manner to a loop of wire carrying a current i. Consider that this loop lies in the x–y plane shown in Figure 1.1, and that the loop is divided into strips such as $abcd$, which will each carry i (this is legitimate since the currents in sides ab and cd will cancel for neighboring strips). There is a magnetic field $\boldsymbol{B}$ (*flux density*) in the magnet, which we shall take to lie in an x–z plane. $\boldsymbol{B}$ may then be resolved into components $B\cos\theta$ in the z direction and $B\sin\theta$ in the x direction, perpendicular and parallel to the plane of the loop respectively. The force due to a magnetic field on a conductor carrying electrical current is known to be proportional to the current (i), flux density (B) and conductor length (l), and the constant of proportionality is the sine of the angle between $\boldsymbol{i}$ and $\boldsymbol{B}$. Thanks to our choice of orientation for the loop and the field, the forces in all four sides ab, bc, cd and da due to $B\cos\theta$ are in equilibrium, and the forces in the long sides ab and cd due to $B\sin\theta$ are zero. However, those in sides bc and da due to $B\sin\theta$ form a couple (or torque) of magnitude

$$\delta \boldsymbol{T} = abi\,\delta y\,B\sin\theta \qquad (1.1)$$

Because the area of the strip is $\delta A = ab\,\delta y$, this becomes

$$\delta \boldsymbol{T} = i\,\delta A\,B\sin\theta \qquad (1.2)$$

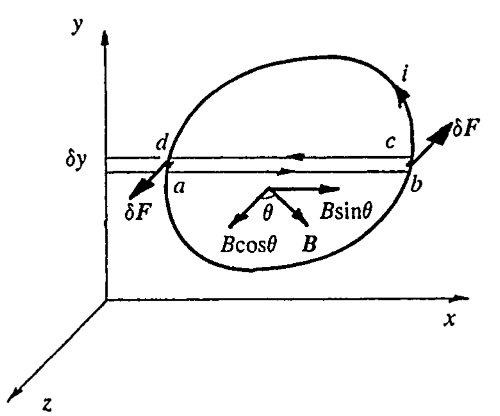


Figure 1.1. A loop carrying current i in field $\boldsymbol{B}$.

Even when modeled as a current loop, the magnetic dipole moment is still a microscopic property of the material. All the strips that comprise the loop in Figure 1.1 carry the same current i, and the entire loop will experience a unique field $\boldsymbol{B}$. The torque on the loop is therefore a summation of Equation (1.2) for all the δA areas that constitute the complete loop $\boldsymbol{A}$, so

$$\boldsymbol{T} = i\boldsymbol{A}B \sin\theta \tag{1.3}$$

The boundary of area $\boldsymbol{A}$ is coincident with the path of i, so it is natural to define their product as a unique parameter, which is the *magnetic dipole moment*:

$$\boldsymbol{\mu}_{\mathrm{m}} = i\boldsymbol{A} \tag{1.4}$$

The torque on the current loop now becomes

$$\boldsymbol{T} = \boldsymbol{\mu}_{\mathrm{m}}B \sin\theta \tag{1.5}$$

Torque $\boldsymbol{T}$, moment $\boldsymbol{\mu}_{\mathrm{m}}$ and the field $\boldsymbol{B}$ (as $B \sin\theta$) are all vector quantities with specific directions as well as magnitudes. No particular shape was defined for the loop in Figure 1.1, so the area $\boldsymbol{A}$ may describe any shape of loop provided that it is planar. The vector $\boldsymbol{A}$ is normal to that plane, so too will be $\boldsymbol{\mu}_{\mathrm{m}}$.

The rectangular current loop of Figure 1.2 has $\boldsymbol{\mu}_{\mathrm{m}}$ normal to its plane, and in the presence of a magnetic field $\boldsymbol{B}$ there is a torque $\boldsymbol{T}$ about its axis. If the loop can rotate about this axis, then the torque tries to align $\boldsymbol{\mu}_{\mathrm{m}}$ with the direction of $\boldsymbol{B}$; if this is achieved, then θ and $\sin\theta$ are zero, as will be $\boldsymbol{T}$.

An electron in a microscopic orbit has a magnetic dipole moment, which we have modeled as a current circulating in a loop. A simple model of a bulk magnetic material is a large array of electrons, and this material will appear to be unmagnetized if its magnetic dipole moments are randomly

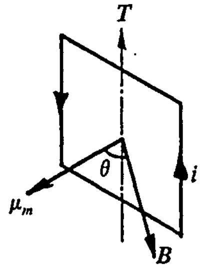


Figure 1.2. Torque applied to a magnetic dipole current loop.

oriented. When a field $\boldsymbol{B}$ is applied to the material, each moment experiences a torque that tends to rotate it towards the direction of $\boldsymbol{B}$. There are atomic forces on the electron orbits, which resist rotation by $\boldsymbol{T}$, but if the applied field is strong enough, then the torques will be great enough to align all the moments with $\boldsymbol{B}$. The material has then reached its *saturation* field. This mechanism is not sufficient, however, to create a permanent magnet material, which by our earlier definition must be able to sustain its own magnetic flux in the absence of any external sources of field. In other words, a permanent magnet must sustain flux by virtue of its own internal field, which will require spontaneous alignment of the magnetic dipole moments, or *spontaneous magnetization.*

The work done in rotating the moment in a field is found by integrating Equation (1.5) through the angle of rotation,

$$E = \int \boldsymbol{T} \cdot \mathrm{d}\theta$$

$$= \mu_m B \int \sin\theta \cdot \mathrm{d}\theta = -\mu_m B \cos\theta \tag{1.6}$$

The lowest value of energy E in the material occurs when μ_m and $\boldsymbol{B}$ are aligned, so there is indeed a natural tendency for this internal alignment to happen and hence minimize the energy. This process of aligning the axes of magnetic dipoles due to their own internal field is called *exchange interaction.* In an elemental volume of the bulk material, the adjacent values (and directions) of μ_m will be identical, so a summation of μ_m may be performed over this volume ΔV, which yields a new property of the material called its *magnetization* $\boldsymbol{M}$:

$$\boldsymbol{M} = \lim_{\Delta V \to 0} \frac{\sum \mu_m}{\Delta V} \tag{1.7}$$

This strict definition of $\boldsymbol{M}$ shows that it is actually the magnetic dipole moment per unit volume, but unlike μ_m, it is a macroscopic property of the material, like $\boldsymbol{B}$. In an elemental volume alone, although the dipoles are aligned by the internal field, that field $\boldsymbol{B}$ will *itself* be generated by $\boldsymbol{M}$. The existence of an internal field even when no external field is applied is the phenomenon of *spontaneous magnetization.* This direct relationship is expressed as $\boldsymbol{B} = \mu_0 \boldsymbol{M}$ (where μ_0 is a constant of proportionality), which allows Equation (1.6) to be rewritten as

$$E = -\mu_0 \mu_m M \cos\theta \tag{1.8}$$

The foregoing is clearly a simplified theory of magnetization, which neglects many of the practical conditions that occur. For example, as we shall see later, thermal agitation will disrupt the alignment of the moments and reduce their magnetization $\boldsymbol{M}$.

1.3 Magnetizing force

While spontaneous magnetization yields a specific value of $\boldsymbol{M}$ in an elemental volume of a magnet, it is not likely that an entire magnet will operate in the same condition with a unique $\boldsymbol{M}$ throughout. A magnet actually comprises a large number of elements ΔV, each of which has a volume $\delta x \cdot \delta y \cdot \delta z$ in Figure 1.3. Two adjacent elements may have different magnitudes for $\boldsymbol{M}$, and we shall consider only the z components of these shown in Figure 1.3, M_z and M_z'. For the element with M_z, Equation (1.7) gives

$$\sum \mu_{mz} = M_z \, \delta x \, \delta y \, \delta z \tag{1.9}$$

Since μ_{mz} is constant over an elemental volume, Equation (1.4) may be used to give

$$\sum \mu_{mz} = i \, \delta x \, \delta y \tag{1.10}$$

Considering that $\boldsymbol{M}$ was defined directly from μ_m, it is hardly surprising to find that $\boldsymbol{M}$ in an element is related to a circulating current i, as shown in Figure 1.3. Combining Equations (1.9) and (1.10), that relationship is

$$i = M_z \, \delta z \tag{1.11}$$

Similarly, in the neighboring element with M_z' there is

$$i' = M_z' \, \delta z \tag{1.12}$$

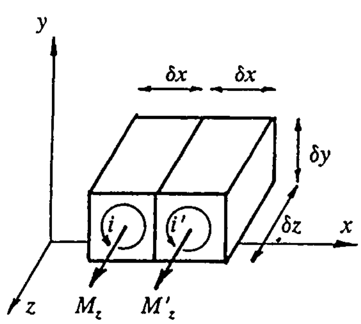


Figure 1.3. Magnetization in neighboring magnet elements.

The change from M_z to M'_z occurs over a distance δx, so the corresponding change in current may be expressed as

$$i' = i + \frac{\partial i}{\partial x}\,\delta x \tag{1.13}$$

With the aid of Equation (1.11), this current flowing up the wall that divides the two elements may be written as

$$i - i' = -\frac{\partial M_z}{\partial x}\,\delta x\,\delta z \tag{1.14}$$

This "wall" may be thought of as having an area $\delta x \cdot \delta z$ across which the current $i - i'$ flows in the y direction. This is a current density, given by

$$J_y = -\frac{\partial M_z}{\partial x} \tag{1.15}$$

Now let us consider that $\boldsymbol{M}$ has a component in the x direction as well as that in the z direction. A change in M_x will also contribute to J_y (although a y component of $\boldsymbol{M}$ would not produce an equivalent current in its own direction). A similar derivation gives

$$J_y = +\frac{\partial M_x}{\partial z} \tag{1.16}$$

The total y component of current density will be

$$J_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x} \tag{1.17}$$

According to this expression, it is a change in the magnetization that is equivalent to the flow of current. J_y is just one of the three possible components of $\boldsymbol{J}_m$ (J_x and J_z are the others), the current density which may be used to model $\boldsymbol{M}$. A full derivation of $\boldsymbol{J}_m$ yields a vector equation, which separates these three components using unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ in the x, y and z directions respectively:

$$\boldsymbol{J}_m = \boldsymbol{i}\left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}\right) \tag{1.18}$$

The $\boldsymbol{j}$ component in the y direction is the one that was derived in Equation (1.17). Fortunately, there is a shorthand notation for the operation that is performed upon a vector such as $\boldsymbol{M}$ in Equation (1.18), to yield another vector such as $\boldsymbol{J}_m$. It is the *curl* operation, sometimes referred to as *rotation*, which is written mathematically as "$\nabla \times$" (Shercliff, 1977). Equation (1.18) is therefore written as

$$J_m = \nabla \times M \tag{1.19}$$

As an example of this operation, imagine that there is a *rotation* of the M vector in the magnet, which causes the M_z component to change between adjacent elements, as was shown in Figure 1.3. The result is an equivalent current density, which we found by Equation (1.15) to be J_y, but which we now know to be just one component from the more general expressions (1.18) and (1.19).

If the magnet is *uniformly* magnetized, there will only be a change in M at the boundaries as shown in Figure 1.4, and there is only an effective rotation of M at the sides, which is where the equivalent current density J_m will flow. A uniformly magnetized magnet is therefore exactly equivalent to a solenoid coil, as one might indeed expect.

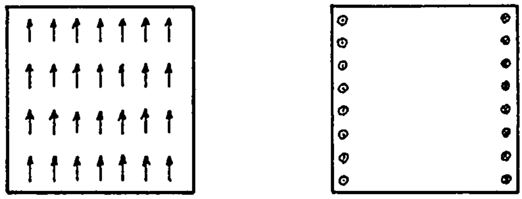
Another well-known situation in electromagnetics to which the *curl* operation applies is the magnetic field B circulating (rotating) around a conductor carrying current – *Ampère's law* (Shercliff, 1977). B is related to the current density J through the constant μ_0 by

$$\mu_0 J = \nabla \times B \tag{1.20}$$

If J is only in the y direction with just a j component as shown in Figure 1.5, then J_y will have the same form as Equation (1.17):

$$\mu_0 J_y = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \tag{1.21}$$

As a *real* current density J causes a field B to circulate around it, then so too does an *equivalent* current density J_m representing magnetization. When both conducting and magnetic media are present, the total flux



(a) (b)

Figure 1.4. Uniformly magnetized magnet (*a*), which may be modeled by a current density over its boundary (*b*).

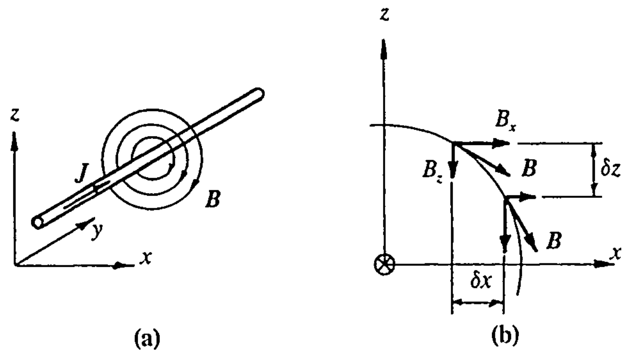


Figure 1.5. Circulation of flux density $\boldsymbol{B}$ around a conductor carrying current density $\boldsymbol{J}$.

density due to both sources will be

$$\mu_0(\boldsymbol{J}+\boldsymbol{J}_{\mathrm{m}})=\nabla\times\boldsymbol{B} \tag{1.22}$$

However, $\boldsymbol{J}_{\mathrm{m}}$ itself is caused by the material magnetization $\boldsymbol{M}$ according to Equation (1.19), so

$$\mu_0(\boldsymbol{J}+\nabla\times\boldsymbol{M})=\nabla\times\boldsymbol{B} \tag{1.23}$$

This expression may be rearranged to group the two magnetic parameters together as

$$\mu_0\boldsymbol{J}=\nabla\times(\boldsymbol{B}-\mu_0\boldsymbol{M}) \tag{1.24}$$

In the previous Section we showed that *spontaneous magnetization* could alone cause a field within the material given by $\boldsymbol{B}=\mu_0\boldsymbol{M}$, which is the condition in Equation (1.24) for *real* currents to be absent ($\boldsymbol{J}=0$). In reality, we will be interested in the *external* effects that a permanent magnet can produce for us, which will of necessity require $\boldsymbol{B}$ to be dissimilar to $\mu_0\boldsymbol{M}$. To this end, a new parameter called *magnetizing force* $\boldsymbol{H}$ is defined as

$$\mu_0\boldsymbol{H}=\boldsymbol{B}-\mu_0\boldsymbol{M} \tag{1.25}$$

The more common way to write this most fundamental relationship between the three macroscopic parameters of a magnet is

$$\boldsymbol{B}=\mu_0(\boldsymbol{H}+\boldsymbol{M}) \tag{1.26}$$

This allows Equation (1.24) to be rewritten as a more general version of Ampère's Law, one that allows for real currents and material magnetization:

$$\boldsymbol{J}=\nabla\times\boldsymbol{H} \tag{1.27}$$

In a magnetic material, an applied current (density $\boldsymbol{J}$) actually causes $\boldsymbol{H}$, rather than $\boldsymbol{B}$, though for a non-magnetic medium we find that Equation (1.27) reverts to Equation (1.20) by setting $\boldsymbol{M}=0$ and using $\boldsymbol{B}=\mu_0\boldsymbol{H}$. If a winding is placed around a permanent magnet and electrical current is applied, then the magnet becomes magnetized by virtue of $\boldsymbol{J}$ establishing $\boldsymbol{H}$ within the material. $\boldsymbol{H}$ in turn establishes $\boldsymbol{M}$ and hence $\boldsymbol{B}$, which explains it being called *magnetizing force*.

1.4 Magnetocrystalline anisotropy

Many permanent magnet materials are manufactured in a way that enhances their magnetic properties along a preferred axis, because in most applications we are only interested in field being produced in one particular direction through the magnet. The most fundamental way to attain this is if the crystal lattice structure of the material itself has preferred directions for the magnetic moments, which may then form the foundation for achieving net alignment in the magnet. Such an alignment of the magnetic dipole moments in the lattice is called *magnetocrystalline anisotropy*.

It is already seen in Equation (1.8) that the work done in rotating $\boldsymbol{\mu}_m$ in a material with magnetization $\boldsymbol{M}$ would have a minimum when $\boldsymbol{\mu}_m$ and $\boldsymbol{M}$ are aligned, and we can use this relationship as the basis for determining the preferred axes in a crystal lattice. It is helpful to rewrite Equation (1.8) using a trigonometric identity for $\cos\theta$ as

$$E=-\mu_0\mu_m M\left(1-2\sin^2\frac{\theta}{2}\right) \tag{1.28}$$

We can now define a *magnetocrystalline anisotropy energy* E_k as the *change* in energy that is required to rotate $\boldsymbol{\mu}_m$ from a preferred axis ($\theta=0$):

$$E_k=2\mu_0\mu_m M\left(\sin^2\frac{\theta}{2}\right) \tag{1.29}$$

In this expression, E_k is a maximum when $\theta=\pi$ (180°) and $\boldsymbol{\mu}_m$ is anti-parallel to $\boldsymbol{M}$ – an *unstable* condition.

The crystal lattices of real magnetic materials are much more complicated than this situation depicts, although Equation (1.29) can be easily modified to account for greater complexity. For example, iron, which is the principal element in many popular permanent magnets, has the body-centered *cubic* crystal lattice structure shown in Figure 1.6. There are now six equally preferred directions of magnetization: [0, 0, 1], [0, 1, 0], [1, 0, 0],

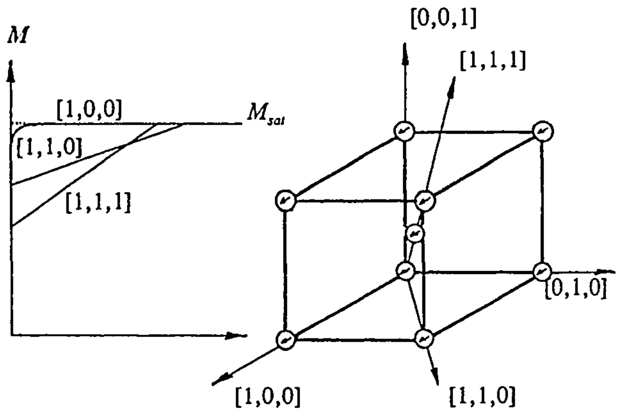


Figure 1.6. Body-centered cubic lattice structure of iron, and magnetization curves on various crystallographic axes.

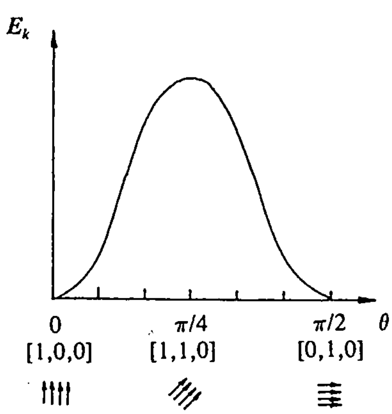


Figure 1.7. Magnetocrystalline anisotropy energy E_k in a cubic crystal lattice structure.

[0, 0, −1], [0, −1, 0] and [−1, 0, 0]. The *unstable* condition for μ_m is to lie in a plane between two of these at an angle of $\pi/4$ (45°) to each axis. It is only necessary to increase the periodicity in Equation (1.29) for it to apply to iron, as

$$E_k = 2\mu_0 \mu_m M \sin^2 2\theta \tag{1.30}$$

The plot of Equation (1.30) shown in Figure 1.7 demonstrates the stability of μ_m along two of the axes in one plane. In reality, rotation of μ_m may be considered in many other planes of the body-centered cube,