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# Preface

This small book grew out of a desire to perceive the hierarchy of sets as an algebraic structure of a simple kind. We began our work on the hierarchy and on small maps around 1988, simultaneously with related work on open maps (see our (1990), (1994) in the bibliography). The main results were announced in our (1991), and presented in several lectures between 1990 and 1993.

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