Introduction

In this book we present a formalization of set theory based on operations on sets, rather than on properties of the membership relation. The two operations are union and successor (singleton), and the algebras for these operations will be called Zermelo-Fraenkel algebras. The definition of these algebras uses an abstract notion of "small map". We show that the usual axioms of Zermelo-Fraenkel set theory are nothing but a description of the free ZF-algebra, just as the axioms of Peano arithmetic form a description of the free monoid on one generator.

The basic ideas are quite simple, and could roughly be explained as follows. Imagine a "universe of sets" $\mathcal{C}$, in which one distinguishes some sets as "small". For example, one could take for $\mathcal{C}$ all the countable sets, and call a set small if it is finite. Another example is provided by taking for $\mathcal{C}$ all the classes (in the sense of set theory), and calling a class small if it is a set, rather than a proper class. In such a universe $\mathcal{C}$ we consider partially ordered sets L which have the property that each "small" subset of L has a supremum, and which are equipped with a distinguished operation $s : L \to L$, called successor. Thus L could be thought of as an algebraic structure, with rather a lot of operations: besides the successor s, which is a unary operation, there is for each small set $S \in \mathcal{C}$ an S-ary operation on L, given by the S-indexed supremum.

In spite of this multitude of operations, it is possible to apply many constructions and results from algebra to such "algebras" L. In fact, they are not so different from, for example, the standard differential algebras: our algebras have small sups instead of finite sums, and a successor s instead of a differential d; they do not satisfy the identity $d \circ d = 0$ of differential algebra, but we shall consider other identities for the successor s.

For two algebras L and L', a homomorphism from L to L' is of course a mapping $\varphi : L \to L'$ which commutes with the operations. In other words, φ preserves small suprema, and commutes with the successor. With these homomorphisms, one can apply the usual constructions of algebra by "generators and relations". For example, the free algebra on a set A, which we denote by $V(A)$, is uniquely defined by the property that for any algebra L and any mapping $A \to L$, there is a uniquely defined extension to a homomorphism $V(A) \to L$. The relation to formal (Zermelo-Fraenkel) set theory now becomes apparent: in the example, mentioned above, where $\mathcal{C}$ consists of all the classes, $V(A)$ is essentially the cumulative hierarchy of sets built on A as a collection of atoms. In the example above where $\mathcal{C}$ consists of countable sets and "small" means finite, the free algebra $V(\emptyset)$ on the empty set is the algebra of hereditarily finite sets. A typical example of adding a "relation" is the algebra O, freely generated by the condition that the successor is monotone. In the example where $\mathcal{C}$ consists of all the classes, O is the class of ordinal numbers. In the example where $\mathcal{C}$ consists of countable sets, O is the set of natural numbers. The algebraic properties of O capture transfinite induction for ordinals, and ordinary induction for natural numbers, respectively.

Of course, the theory can only be developed when one assumes that the collection of "small" sets satisfies some suitable axioms. To get off the ground at all, we will assume that the empty set and the one-point set are both small. Thus, in particular, any algebra L contains the supremum of the empty subset of L; i.e. L must have a smallest element 0. Furthermore, we will assume that the union of a small family of small sets is small, and that the disjoint sum of two small sets is small. Thus, in particular, the two-point set is small, and hence any algebra L has an operation $\vee : L \times L \to L$ of binary supremum. There are also axioms for covers (i.e., surjections): if $S \twoheadrightarrow T$ is a cover and S is small then T is small; and conversely, if T is small then S contains a small subset $S' \subseteq S$ which already covers T. In the example where $\mathcal{C}$ consists of all classes, this latter property for the existence of "small subcovers" is usually referred to as the *collection axiom* of set theory. Finally, the following two axioms play a crucial rôle in the construction of algebras by generators and relations. First, for any set C in $\mathcal{C}$ and any small set S in $\mathcal{C}$, one can form the set (not necessarily small) C^S of all functions from S to C. Secondly, there exists a "universal" small set in $\mathcal{C}$: this is a mapping $\pi : E \to U$ in $\mathcal{C}$ such that for any small set S there is some point $x \in U$ for which S is isomorphic to $\pi^{-1}(x)$. These are essentially all the axioms that we will ask the collection of "small" sets in the universe $\mathcal{C}$ to satisfy.

It is important to observe that it is only necessary for the "universe of sets" $\mathcal{C}$ in which we construct and study the algebras L to possess some very basic properties. It should be possible to interpret the basic operations of the first order logic (conjunction, disjunction, universal and existential quantification, etc.) in $\mathcal{C}$. This means, for example, that for two mappings $\alpha : S \to T$ and $\beta : R \to S$ in $\mathcal{C}$ one can form the set $\{t \in T \mid \forall s \in S :$

if $\alpha(s) = t$ then $\exists\, r \in R\ :\ \beta(r) = s\}$ inside the universe $\mathcal{C}$. These logical operations need not even satisfy the rules of classical first order logic; but they should satisfy at least the rules of intuitionistic logic. There are many interesting examples of such universes $\mathcal{C}$ which are quite different from the usual universe of sets. Thus, $\mathcal{C}$ can be a universe of *sheaves* (i.e., sets which vary continuously over a fixed topological space of parameters), or an "effective" universe of *recursive* sets. More generally, $\mathcal{C}$ can be any *elementary topos* (see Mac Lane-Moerdijk (1992) for this notion and many examples of elementary topoi). In this way, our theory extends both topos theory and (intuitionistic) set theory. In particular, the theory is powerful enough to capture in a constructive way the theory of ordinals and of transfinite induction.

In fact, one of our main motivations was the apparent discrepancy between sheaves and topoi on the one hand, and models of Zermelo-Fraenkel set theory on the other. Elementary topoi correspond naturally to a weak kind of set theory with only bounded quantifiers, as discussed extensively in the early topos literature (cf. Mitchell(1972), Cole(1973), Osius(1974), and others). For an arbitrary topos, it is in general not possible to build a corresponding model for Zermelo-Fraenkel set theory. Nevertheless for many topoi which are constructed using set theory to begin with (such as topoi of Boolean sets or of sheaves), one can obtain correponding models for Zermelo-Fraenkel set theory by a transfinite iteration of the power-set operation of the topos along the "external", classical, ordinal numbers. For Boolean sets, this construction goes back to Scott and Solovay (see Bell(1977)). For general sheaves, it is discussed in Fourman(1980), Freyd(1980), Blass-Scedrov(1989), and elsewhere.

To describe the variety of different examples of universes $\mathcal{C}$ having the required properties, and to exploit relations with topos theory, we will formulate our theory using the language of categories (Mac Lane(1971)). For readers who are not sufficiently familiar with this language, we hasten to point out that much of this book can be read and understood at a less general level, by assuming throughout that $\mathcal{C}$ is an actual universe of sets, as in the two examples – countable sets and classes – mentioned above.

In the language of category theory, the "small" objects in $\mathcal{C}$ will (have to) be described in terms of small maps. Intuitively, these are the maps $f : E \to X$ all of whose fibers $f^{-1}(x)$ are small; a small map $E \to X$ in $\mathcal{C}$ can be thought of as a continuous family of small objects, parametrized by X. A basic axiom for these small maps, which expresses that smallness is a property of the fibers of the map, is that in a pullback square

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X, \end{array}$$

the map $E' \to X'$ is small whenever $E \to X$ is, while conversely, if $E' \to X'$ is small and $X' \to X$ is surjective then $E \to X$ is small. Other axioms are direct translations of the axioms for small sets mentioned before. For example, the axiom that a small union of small sets is small can now simply be expressed by stating that the composition of two small maps is again small.

The idea of continuous families of "small" objects, constructed as mappings $E \to X$ with suitable properties, is ubiquitous in geometry and physics. Well-known examples include live bundles (families of lines) and proper maps (families of compact spaces) in topology, and families of curves in algebraic geometry. In this context, one often studies universal families of such small objects, such as classifying spaces for line bundles (projective spaces and Grassmann manifolds) and moduli spaces of curves. When stated for small maps, our axiom for a universal small set takes a similar form: it states that there is an object U with a small map $\pi : E \to U$, such that every small map is locally a pullback of this universal small map $\pi : E \to U$. Thus, U is a "classifying space" for small maps, having properties much like classifying spaces for vector bundles and other well-known classifying spaces in topology. (To illustrate the analogy, we explain at the end of Chapter I how the classifying space U for small maps is "unique up to homotopy".)

Our abstract framework thus consists of a suitable category $\mathcal{C}$, with a designated class of arrows in $\mathcal{C}$, which are called small, and satisfy natural axioms. In this general context, it is possible to define algebras L as objects in $\mathcal{C}$ equipped with an operation $s : L \to L$ for successor, and with a partial order on L which is complete in the sense that the supremum exists along any map which is designated as small. Such algebras L will be called *Zermelo-Fraenkel algebras* in $\mathcal{C}$. We investigate the structure of the free (initial) ZF-algebra V, and show that it can be viewed as an algebra of small sets, via an explicit isomorphism between V and the object $P_s(V)$ of "small subsets" of V. This free algebra V should be viewed as the cumulative hierarchy of small sets, relative to the ambient category $\mathcal{C}$ and its class of small maps. Indeed, we prove in Chapter II, §5, that under very general conditions this algebra V is a model of the axioms for (intuitionistic) Zermelo-Fraenkel set theory. In Chapter IV, we will explain in detail how one obtains, as particular examples, the sheaf models and effective (realizability) models for set theory already referred to above.

Our algebraic approach also makes it possible to distinguish different types of ordinal numbers in a very natural way. For example, in Chapter II, §2, we will discuss how, within the category $\mathcal{C}$, the ZF-algebra O which is free on a monotone successor operation $t : O \to O$ enables one to write V as a cumulative hierarchy of objects V_α, suitably indexed by elements $\alpha \in O$. The classical Von Neumann ordinals, defined as hereditarily transitive sets, also appear as a free algebra, generated by the relation that the successor is inflationary ($x \leq s(x)$). Furthermore, in Chapter II, §4, it will be discussed how the ZF-algebra T which is free on a successor $r : T \to T$ preserving binary suprema enables one to give a purely constructive proof of Tarski's fixed point theorem, using "transfinite induction" along this object T.

In Chapter III, it will be shown how all these free algebras can be explicitly constructed as objects of the ambient category $\mathcal{C}$. This construction of free algebras makes use of the theory of open maps, and of (bi-)simulations for trees and forests. The explicit use of bisimulation for set theory goes back to the work on non-well-founded sets by Aczel(1988). It would be of interest to construct sheaf models for the theory of non-well-founded sets from our axioms for small maps.

Cambridge University Press
0521558301 - Algebraic Set Theory
A. Joyal and I. Moerdijk
Excerpt
More information

Chapter I

Axiomatic Theory of Small Maps

§1 Axioms for small maps

In this first section we will present a set of axioms for a class $\mathcal{S}$ of small maps in a category $\mathcal{C}$. These axioms are meant to express some basic properties of maps with "small" fibers. For example, if $\mathcal{C}$ is the category of sets, our axioms are satisfied by the class of maps with finite fibers (those $f : Y \to X$ with $f^{-1}(x)$ finite for each $x \in X$), or the class of maps with countable fibers, etc.

The ambient category $\mathcal{C}$ will be assumed to be a Heyting pretopos with a natural numbers object. This means that $\mathcal{C}$ is a category with enough structure to interpret (intuitionistic) first order logic and arithmetic. (We recall the precise definition in Appendix B.)

Our axioms for small maps are an extension of the axioms for open maps presented in Joyal-Moerdijk(1990) and (1994), and we begin by recalling those. Consider the following properties of a class $\mathcal{S}$ of arrows in the category $\mathcal{C}$.

(A1) Any isomorphism belongs to $\mathcal{S}$, and $\mathcal{S}$ is closed under composition.

(A2) ("Stability") In any pullback square

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[p]{} & X \end{array} \qquad (1)$$

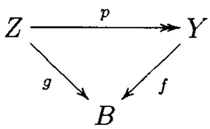
if f belongs to $\mathcal{S}$ then so does g.

(A3) ("Descent") In any pullback square (1), if g belongs to $\mathcal{S}$ and p is epi then f belongs to $\mathcal{S}$.

(A4) The maps $0 \to 1$ and $1 + 1 \to 1$ belong to $\mathcal{S}$.

(A5) ("Sums") If $Y \to X$ and $Y' \to X'$ belong to $\mathcal{S}$ then so does their sum $Y + Y' \to X + X'$.

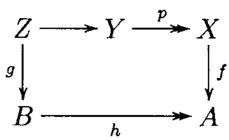
(A6) ("Quotients") In any commutative diagram



(2)

if p is epi and g belongs to $\mathcal{S}$ then so does f.

(A7) ("Collection Axiom") For any two arrows $p : Y \twoheadrightarrow X$ and $f : X \to A$ where p is epi and f belongs to $\mathcal{S}$, there exists a quasi-pullback square of the form



(3)

where h is epi and g belongs to $\mathcal{S}$.

(Recall that such a square is said to be a quasi-pullback if the obvious arrow $Z \to B \times_A X$ is an epimorphism.)

The class $\mathcal{S}$ is said to be a class of *open maps* (with collection) if it satisfies these axioms (A1 – 7). For standard examples of such classes we refer the reader to Joyal-Moerdijk(1994).

Before we state our axioms for a class of small maps, we recall that a map $f : Y \to X$ in $\mathcal{C}$ is said to be *exponentiable* if f is exponentiable as an object of the slice category $\mathcal{C}/X$ (i.e., the functor $f^* : \mathcal{C}/X \to \mathcal{C}/X$ sending $Z \to X$ to $Z \times_X Y \to X$ has a right adjoint).

1.1 Definition. A class $\mathcal{S}$ of arrows in a category $\mathcal{C}$ is said to be a *class of small maps* if $\mathcal{S}$ is a class of open maps (with collection) satisfying the following two additional axioms (S1) and (S2).

(S1) ("Exponentiability Axiom") Every map in $\mathcal{S}$ is exponentiable.

(S2) ("Representability Axiom") There exists a map $\pi : E \to U$ in $\mathcal{S}$ which is universal in the following sense: for any map $f : Y \to X$ in $\mathcal{S}$ there exists a diagram

$$\begin{array}{ccccc} Y & \longleftarrow & Y' & \longrightarrow & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \pi} \\ X & \underset{p}{\twoheadleftarrow} & X' & \underset{c}{\longrightarrow} & U \end{array} \qquad (4)$$

in which p is epi and both squares are pullbacks.

Note that this Representability Axiom states that every map in $\mathcal{S}$ is "locally" a pullback of the universal map $\pi : E \to U$.

From now on, we will refer to a map $f : Y \to X$ in $\mathcal{S}$ as a "small map", or as a "small object over X". Furthermore, an object Y of $\mathcal{C}$ is said to be "small" if the unique map $Y \to 1$ belongs to $\mathcal{S}$.

In the rest of this section, we will make some elementary first observations concerning these axioms for small maps. First notice the following closure properties for exponentiable maps.

1.2 Lemma. *In any Heyting pretopos $\mathcal{C}$, the class of exponentiable maps satisfies the axioms (A1 - 6) for open maps.*

Proof. Write $\mathcal{E}$ for the class of exponentiable maps. First recall that a map $f : Y \to X$ is exponentiable iff the pullback functor $f^* : \mathcal{C}/X \to \mathcal{C}/Y$ has a right adjoint Π_f. From this it is clear that the class $\mathcal{E}$ satisfies axiom (A1). Axioms (A4) and (A5) also clearly hold, the exponential of a sum being constructed as a product (of exponentials). For (A6), it suffices to consider the case $B = 1$ (replace $\mathcal{C}$ by $\mathcal{C}/B$). But for any epimorphism $p : Y \twoheadrightarrow X$, any exponential A^X can be constructed from the exponential A^Y using the universal quantifiers in $\mathcal{C}$, as $A^X = \{f \in A^Y | \forall y_1, y_2 \in Y(p(y_1) = p(y_2) \Rightarrow f(y_1) = f(y_2))\}$. For (A2) assume again that $X = 1$. Then for any exponentiable object Y, any exponential $(A \to X')^{(Y \times X' \to X')}$ in $\mathcal{C}/X'$ can be constructed from the transpose $X' \to X'^Y$ of the projection, as the pullback

$$\begin{array}{ccc} \bullet & \longrightarrow & A^Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X'^Y. \end{array}$$

Finally, we outline the proof for the descent axiom (A3). We may assume again that $X = 1$. Suppose $X' \twoheadrightarrow 1$ is epi and $Y \times X' \to X'$ is exponentiable in $\mathcal{C}/X'$. Consider an object $A \in \mathcal{C}$, and denote the exponential $(A \times X' \to X')^{(Y \times X' \to X')}$ by $E \to X'$. By axiom (A2), already verified, it follows for the two projections π_1 and $\pi_2 : X' \times X' \rightrightarrows X'$ that the two pullbacks $\pi_i^*(Y \times X' \to X')$ are exponentiable in $\mathcal{C}/X' \times X'$, with exponential $\pi_i^*(E) = \pi_i^*(A \times X' \to X')^{\pi_i^*(Y \times X' \to X')}$ (for $i = 1, 2$). It follows that $E \to X'$ is equipped with canonical descent data. Since in a pretopos every epi is an effective descent map (see Appendix C), it follows that $E \to X'$ is isomorphic to a projection $D \times X' \to X'$, for an object $D \in \mathcal{C}$ uniquely determined up to isomorphism. It is now routine to verify that D is the exponential A^Y.

1.3 Remark. It follows from Lemma 1.2 that the axioms (S1) and (S2) for small maps are equivalent to the single axiom stating that there exists a small map $\pi : E \to U$ which is universal (as in (S2)) as well as exponentiable.

1.4 Remark. Observe that $\pi : E \to U$ is not unique, nor is (given π) the characteristic map c in (4). In fact, for a pullback square

$$\begin{array}{ccc} E & \longrightarrow & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ U & \xrightarrow[f]{} & U' \end{array}$$

with f epi, π is universal iff π' is. However, there is a uniqueness up to "homotopy", just as for universal vector bundles and similar constructions in topology. We refer to the appendix in this chapter (§5) for a precise formulation.

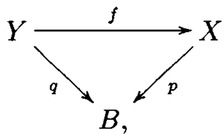
Next, we note the stability of our axioms under slicing. For this, let $\mathcal{S}$ be a class of small maps in $\mathcal{C}$, and let B be an object of $\mathcal{C}$. Define an induced class $\mathcal{S}_B$ in the slice category $\mathcal{C}/B$ in the obvious way: writing $\sum_B : \mathcal{C}/B \to \mathcal{C}$ for the forgetful functor, a map f in $\mathcal{C}/B$ belongs to $\mathcal{S}_B$ iff $\sum_B(f)$ belongs to $\mathcal{S}$. In the proposition below, $B^* : \mathcal{C} \to \mathcal{C}/B$ denotes the functor $X \mapsto (\pi_2 : X \times B \to B)$, right adjoint to $\sum_B$.

1.5 Proposition. *Let $\mathcal{S}$ be a class of small maps in $\mathcal{C}$. Then $\mathcal{S}_B$ is a class of small maps in $\mathcal{C}/B$; moreover, the functor $B^* : \mathcal{C} \to \mathcal{C}/B$ preserves small maps, as well as the universal small map.*

Proof. The fact that $\mathcal{S}_B$ satisfies the axioms for open maps is a matter of elementary verification. Furthermore, $\mathcal{S}_B$ clearly satisfies the exponentiability axiom (S1), while (S2) holds for $\mathcal{S}_B$ with as universal map $\pi \times B : E \times B \to U \times B$ over B.

To conclude this section, we prove that the notion of "small map" is definable, in the precise sense of the following proposition. As a consequence, one can use the predicate "small" as part of the internal logic of $\mathcal{C}$, as we will freely do in subsequent sections.

1.6 Proposition. *For any arrow f in $\mathcal{C}$ over a base B,*



there exists a subobject $S \rightarrowtail B$ such that for any map $\alpha : C \to B$ in $\mathcal{C}$, the pullback $f \times_B C : Y \times_B C \to X \times_B C$ belongs to $\mathcal{S}$ iff α factors through S.

This object S will be denoted

$$S = \{b \in B \mid f_b : Y_b \to X_b \text{ is small}\}.$$

Proof. Using exponentiability of the universal small map $\pi : E \to U$, the desired object S can be constructed in terms of the first order logic of $\mathcal{C}$, as

$$S = \{b \in B \mid \forall x \in p^{-1}(b) \ \ \exists u \in U \ \ \exists \alpha \in f^{-1}(x)^{\pi^{-1}(u)} : \alpha \text{ is an isomorphism}\}.$$

The verification that this object S has the desired property, stated in the proposition, is straightforward (for example, by using the so-called Kripke-Joyal semantics in $\mathcal{C}$).

§2 Representable structures

In this section we will give some examples of "universal" small structures , obtained from a universal small map $\pi : E \to U$.