1
Governing Equations

All analyses concerning the motion of compressible fluids must necessarily begin, either directly or indirectly, with the statements of the four basic physical laws governing such motions.

A. Shapiro (1953)

1.1 Introduction

This chapter presents the governing equations of one-dimensional unsteady flow of a compressible fluid without derivation.† The following assumptions are made. First, the fluid is assumed to be calorically perfect, *i.e.,* the specific heats c_v and c_p at constant volume and pressure, respectively, are constant. Thus, the internal energy per unit mass e_i is

$$e_i = c_v T \tag{1.1}$$

where T is the static temperature. The static enthalpy h is defined as

$$h = e_i + \frac{p}{\rho} \tag{1.2}$$

Second, the fluid is assumed to be thermally perfect,

$$p = \rho R T \tag{1.3}$$

where p is the static pressure, ρ is the density, and R is the species gas constant,

$$R = c_p - c_v \tag{1.4}$$

† The full equations of three-dimensional compressible viscous flow are presented, for example, in Schreier (1982) and White (1974) .

It follows that

$$h = c_p T \tag{1.5}$$

Third, the fluid is assumed to be inviscid. Fourth, radiation effects and chemical reactions are omitted, and the fluid is assumed to be homogeneous (*i.e.*, uniform molecular composition).

1.2 Conservation Laws

Consider one-dimensional, inviscid unsteady flow in a tube of constant cross-sectional area A. We define a control volume V as shown in Fig. 1.1.

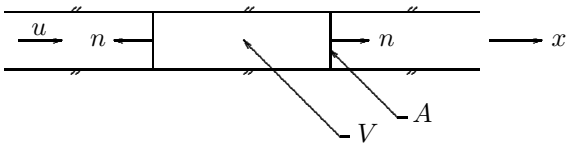

Fig. 1.1. Control volume

The integral conservation equations for mass, momentum, and energy are

$$\frac{d}{dt}\int_V \rho\, dV + \int_A \rho u n dA = 0 \tag{1.6}$$

$$\frac{d}{dt}\int_V \rho u dV + \int_A \left(\rho u^2 + p\right) n dA = 0 \tag{1.7}$$

$$\frac{d}{dt}\int_V \rho e dV + \int_A \left(\rho e + p\right) u n dA = 0 \tag{1.8}$$

where n is the unit vector in the outward direction† and the total energy e is

$$e = e_i + \tfrac{1}{2}u^2 \tag{1.9}$$

The differential forms of the conservation laws are

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{1.10}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} = -\frac{\partial p}{\partial x} \tag{1.11}$$

$$\frac{\partial \rho e}{\partial t} + \frac{\partial \left(\rho e + p\right) u}{\partial x} = 0 \tag{1.12}$$

† On the left face $n = -1$, and on the right face $n = +1$.

1.3 Convective Derivative

The convective derivative of a function f is defined as

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} \tag{1.13}$$

and represents the rate of change of the variable f with respect to time while following a fluid particle. The differential form of the conservation laws for momentum and energy can be rewritten in terms of the convective derivative using the conservation of mass,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{1.14}$$

to obtain

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} \tag{1.15}$$

$$\rho\left(\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x}\right) = -\frac{\partial pu}{\partial x} \tag{1.16}$$

1.4 Vector Notation

The differential form of the conservation laws (1.10) to (1.12) may be written in a compact vector notation as

$$\frac{\partial \mathcal{Q}}{\partial t} + \frac{\partial \mathcal{F}}{\partial x} = 0 \tag{1.17}$$

where

$$\mathcal{Q} = \left\{ \begin{array}{c} \rho \\ \rho u \\ \rho e \end{array} \right\} \tag{1.18}$$

$$\mathcal{F} = \left\{ \begin{array}{c} \rho u \\ \rho u^2 + p \\ \rho e u + p u \end{array} \right\} \tag{1.19}$$

1.5 Entropy

The change in entropy per unit mass s is

$$Tds = de_i - \frac{p}{\rho^2}d\rho \tag{1.20}$$

For a thermally perfect gas, this equation may be integrated to obtain

$$s - s_1 = c_v \ln \frac{T}{T_1} - R \ln \frac{\rho}{\rho_1} \tag{1.21}$$

Using (1.3), two alternate forms may be obtained:

$$s - s_1 = c_p \ln \frac{T}{T_1} - R \ln \frac{p}{p_1} \tag{1.22}$$

$$s - s_1 = c_v \ln \frac{p}{p_1} - c_p \ln \frac{\rho}{\rho_1} \tag{1.23}$$

Therefore, for isentropic flow between two states,

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \tag{1.24}$$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \tag{1.25}$$

The Second Law of Thermodynamics (Shapiro, 1953) may be expressed as

$$dS \geq \frac{\delta Q}{T} \tag{1.26}$$

where S is the entropy of a system (*i.e.*, an identifiable mass of fluid), δQ is the heat added to the system, and T is the static temperature. In particular, this implies that $dS \geq 0$ for an adiabatic process.

1.6 Speed of Sound

The speed of sound is the velocity of propagation of an infinitesimal disturbance in a quiescent fluid and is defined by

$$a = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s} \tag{1.27}$$

where the partial derivative is taken at constant entropy s. For an ideal gas,

$$a = \sqrt{\gamma R T} \tag{1.28}$$

The Mach number is the ratio of the flow velocity to the speed of sound:

$$M = \frac{|u|}{a} \tag{1.29}$$

1.7 Alternate Forms

The total enthalpy H is

$$H = e + \frac{p}{\rho} \tag{1.30}$$

The conservation of energy may be expressed as an equation for the total enthalpy:

$$\frac{\partial \rho H}{\partial t} + \frac{\partial \rho H u}{\partial x} = \frac{\partial p}{\partial t} \tag{1.31}$$

Alternately, the energy and momentum equations may be utilized to obtain an equation for the internal energy e_i,

$$\frac{\partial \rho e_i}{\partial t} + \frac{\partial \rho e_i u}{\partial x} = -p\frac{\partial u}{\partial x} \tag{1.32}$$

Also, the energy may be rewritten in terms of the entropy. Using (1.20),

$$\frac{\partial \rho s}{\partial t} + \frac{\partial \rho s u}{\partial x} = 0 \tag{1.33}$$

It is noted that equations (1.31) to (1.33) can be rewritten in terms of the convective derivative using (1.10):

$$\rho\left(\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x}\right) = \frac{\partial p}{\partial t} \tag{1.34}$$

$$\rho\left(\frac{\partial e_i}{\partial t} + u\frac{\partial e_i}{\partial x}\right) = -p\frac{\partial u}{\partial x} \tag{1.35}$$

$$\rho\left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x}\right) = 0 \tag{1.36}$$

Exercises

1.1 Derive the alternate form of the momentum equation (1.15) using (1.11) and (1.10).

SOLUTION

The difference between (1.11) and (1.15) is the expression on the left side:

$$\begin{aligned}
\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} &= \rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + \rho u\frac{\partial u}{\partial x} + u\frac{\partial \rho u}{\partial x} \\
&= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + u\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x}\right) \\
&= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)
\end{aligned}$$

using (1.10). A similar derivation applies to (1.16).

1.2 Derive the enthalpy equation (1.31).

1.3 Derive the internal energy equation (1.32).

SOLUTION

Multiply the momentum equation (1.15) by u:

$$\begin{aligned} \rho u\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right) &= -u\frac{\partial p}{\partial x} \\ \rho\left[\frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2\right)+u\frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2\right)\right] &= -u\frac{\partial p}{\partial x} \end{aligned}$$

which represents an equation for the kinetic energy† per unit mass $\frac{1}{2}u^2$. Subtract from the energy equation (1.16) using (1.9) to yield

$$\begin{aligned} \rho\left(\frac{\partial e_i}{\partial t}+u\frac{\partial e_i}{\partial x}\right) &= \frac{\partial}{\partial x}(-pu)+u\frac{\partial p}{\partial x} \\ &= -u\frac{\partial p}{\partial x}-p\frac{\partial u}{\partial x}+u\frac{\partial p}{\partial x} \\ &= -p\frac{\partial u}{\partial x} \end{aligned}$$

Multiply the mass equation (1.10) by e_i and add to the above to yield

$$\frac{\partial \rho e_i}{\partial t}+\frac{\partial \rho e_i u}{\partial x} = -p\frac{\partial u}{\partial x}$$

1.4 Derive the entropy equation (1.33).

1.5 Derive a conservation equation for the static enthalpy h.

SOLUTION

Add $\partial p/\partial t$ to both sides of the energy equation (1.12) to yield

$$\frac{\partial \rho H}{\partial t}+\frac{\partial \rho H u}{\partial x}=\frac{\partial p}{\partial t}$$

using (1.30). Multiply the mass equation by $\frac{1}{2}u^2$ and add to the kinetic energy equation (see above) to yield

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2\right)+\frac{\partial}{\partial x}\left(\tfrac{1}{2}\rho u^2 u\right) = -u\frac{\partial p}{\partial x}$$

Subtract from the previous equation to yield

$$\frac{\partial \rho h}{\partial t}+\frac{\partial \rho h u}{\partial x} = \frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}$$

Using the mass equation, this may also be written as

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt}$$

† The equation is also known as the *mechanical energy equation*.

1.6 In the presence of a body force per unit mass f, the momentum and energy equations become

$$\begin{aligned} \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} &= -\frac{\partial p}{\partial x} + \rho f \\ \frac{\partial \rho e}{\partial t} + \frac{\partial \left(\rho e + p\right) u}{\partial x} &= \rho f u \end{aligned}$$

The mass equation is unchanged. Show that the total enthalpy equation is

$$\frac{\partial \rho H}{\partial t} + \frac{\partial \rho H u}{\partial x} = \frac{\partial p}{\partial t} + \rho f u$$

1.7 Derive the following equation:

$$\frac{1}{p}\frac{Dp}{Dt} = \frac{1}{c_v}\frac{Ds}{Dt} + \frac{\gamma}{\rho}\frac{D\rho}{Dt}$$

and provide a physical interpretation of each of the terms.

SOLUTION

From the entropy equation (1.23),

$$s - s_1 = c_v \ln \frac{p}{p_1} - c_p \ln \frac{\rho}{\rho_1}$$

Differentiating,

$$\frac{Ds}{Dt} = \frac{c_v}{p}\frac{Dp}{Dt} - \frac{c_p}{\rho}\frac{D\rho}{Dt}$$

Thus,

$$\frac{1}{p}\frac{Dp}{Dt} = \frac{1}{c_v}\frac{Ds}{Dt} + \frac{\gamma}{\rho}\frac{D\rho}{Dt}$$

The term on the left is the normalized rate of change of the static pressure following a fluid particle. The first term on the right is the rate of change of entropy (divided by c_v) following a fluid particle. Thus, an increase in entropy of the fluid particle acts to increase its static pressure. For an inviscid, homogeneous flow, the rate of change of entropy following a fluid particle is zero from (1.36), except when the fluid particle crosses a discontinuity (*i.e.*, at locations where the derivatives of the flow variables are not defined). The second term on the right is the rate of change of the fluid particle density (divided by ρ/γ). Thus, an increase in fluid particle density acts to increase the static pressure.

1.8 The total pressure p_o at a point is defined as the static pressure achieved by bringing a fluid particle at that point to rest isentropically. Similarly, the total temperature T_o at a point is defined as the static temperature achieved by bringing a fluid particle at that point to rest adiabatically. Therefore,

$$\begin{aligned} p_o &= p\left[1 + \frac{(\gamma - 1)}{2}M^2\right]^{\gamma/(\gamma-1)} \\ T_o &= T\left[1 + \frac{(\gamma - 1)}{2}M^2\right] \end{aligned}$$

Thus, the entropy definition (1.25) may be written

$$s - s_1 = c_p \ln \frac{T_o}{T_{o_1}} - R \ln \frac{p_o}{p_{o_1}}$$

Show that

$$\frac{\rho}{\rho_o}\frac{Dp_o}{Dt} = \frac{\partial p}{\partial t} - \rho T_o \frac{Ds}{Dt}$$

where $\rho_o = p_o/RT_o$. Provide a physical interpretation of each of the terms.

1.9 The mechanical energy equation (see Problem 1.3) is

$$\frac{D}{Dt}\left(\tfrac{1}{2}u^2\right) = -\frac{u}{\rho}\frac{\partial p}{\partial x}$$

Provide a physical explanation.

SOLUTION

The left-hand side of the equation is the time rate-of-change of the kinetic energy per unit mass following a fluid particle. The right-hand side is the work done on the unit mass by the pressure gradient. If the pressure increases in the direction of the flow (*i.e.*, $u\partial p/\partial x > 0$, which implies either a) $u > 0$ and $\partial p/\partial x > 0$ or b) $u < 0$ and $\partial p/\partial x < 0$), the kinetic energy decreases because the pressure gradient decelerates the flow. If the pressure decreases in the direction of the flow (*i.e.*, $u\partial p/\partial x < 0$), the kinetic energy increases because the pressure gradient accelerates the flow.

1.10 Derive an equation for the convective derivative of the Gibbs free energy

$$g = h - Ts$$