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LÉVY PROCESSES AND INFINITELY DIVISIBLE DISTRIBUTIONS

Lévy processes are rich mathematical objects and constitute perhaps the most basic class of stochastic processes with a continuous time parameter. This book is intended to provide the reader with comprehensive basic knowledge of Lévy processes, and at the same time serve as an introduction to stochastic processes in general. No specialist knowledge is assumed and proofs are given in detail. Systematic study is made of stable and semi-stable processes, and the author gives special emphasis to the correspondence between Lévy processes and infinitely divisible distributions. All serious students of random phenomena will find that this book has much to offer.

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Ken-iti Sato is Professor Emeritus at Nagoya University, Japan.

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Lévy Processes and Infinitely Divisible Distributions

Corrected Printing with Supplement

KEN-ITI SATO
Nagoya University, Japan

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Preface to the corrected printing with supplement

After the publication of this book in 1999 progress continued in the theory of Lévy processes and infinitely divisible distributions. Of the various directions let me mention two.

1. Fluctuation theory of Lévy processes on the line has been studied in many papers. It is a development of Wiener–Hopf factorizations. The publication of two books, Kyprianou [**303**] and Doney [**103**], provided fresh impetus. Lévy processes without positive jumps were deeply analyzed in this connection such as in Kuznetsov, Kyprianou, and Rivero [**301**]. Following Lamperti [**306**], the relation between selfsimilar (in an extended sense) Markov processes on the positive half line and exponential functionals of Lévy processes on the real line was studied in Bertoin and Yor [**30, 31**] and others. In the study of stable processes Kyprianou, Pardo, and Watson [**304**] combined this line of research and Wiener–Hopf factorizations.

2. A comprehensive treatment of infinitely divisible distributions on the line appeared in the monograph [**503**] by Steutel and van Harn, which discussed a lot of subjects not treated in this book. The analysis of the law of $\int_0^t \mathrm{e}^{B_s+as}\mathrm{d}s$ (exponential functional of Brownian motion with drift) was explored by Yor and others, for example in Matsumoto and Yor [**341**]. Further, the law of $\int_0^\infty \mathrm{e}^{-X_s}\mathrm{d}Y_s$ for a two-dimensional Lévy process $\{(X_t, Y_t)\}$ began to attract attention such as in Lindner and Sato [**321, 322**]. Tail behaviors related to subexponentiality are another subject in one dimension. In higher dimensions Watanabe's results [**563**] on densities of stable distributions opened a new horizon.

Subordinators (increasing Lévy processes) and their applications were described in Bertoin's lecture [**27**]. We can find a unique approach to them in Schilling, Song, and Vondraček [**472**]. On stochastic differential equations based on Lévy processes, Kunita's paper [**299**] and Applebaum's book [**6**] should be mentioned.

In this new printing a Supplement of 30 pages is attached at the end. The purpose of the Supplement is twofold. First, among a great many areas of progress it covers some subjects that I am familiar with (Sections 59, 62, 63, 64, and a part of 57). Second, it includes some materials closely connected with the original ten chapters (Sections 56, 58, 60, 61 and a part of 57). Changes to the text of the first printing are only few, which

I considered necessary. Any addition or deletion was avoided, with the exception of a few inserted lines in Remarks 15.12 and 37.13 and Definition 51.9. Naturally all numberings except reference numbers remain the same. Thus I chose to preserve the contents of the first printing, refraining from partial improvement. Newly added references are restricted to those cited in the Supplement and in this Preface. They are marked with asterisks in the list of references. For readers of the first printing a list of corrections and changes in the ten chapters is posted on my website (http://ksato.jp/).

There is no writing on the history of the study of Lévy processes and infinitely divisible distributions. But Notes at the end of each chapter of this book and my article [**451**] point out some epoch-making works.

I would like to thank Alex Lindner, Makoto Maejima, René Schilling, and Toshiro Watanabe for valuable comments on the published book and on this printing in preparation. Bibliographical remarks by Alex and René were very helpful. Encouragement from the late Hiroshi Tanaka is my cherished memory.

Ken-iti Sato
Nagoya, 2013

Preface to the first printing

Stochastic processes are mathematical models of random phenomena in time evolution. Lévy processes are stochastic processes whose increments in nonoverlapping time intervals are independent and whose increments are stationary in time. Further we assume a weak continuity called stochastic continuity. They constitute a fundamental class of stochastic processes. Brownian motion, Poisson processes, and stable processes are typical Lévy processes. After Paul Lévy's characterization in the 1930s of all processes in this class, many researches have revealed properties of their distributions and behaviors of their sample functions. However, Lévy processes are rich mathematical objects, still furnishing attractive problems of their own. On the other hand, important classes of stochastic processes are obtained as generalizations of the class of Lévy processes. One of them is the class of Markov processes; another is the class of semimartingales. The study of Lévy processes serves as the foundation for the study of stochastic processes.

Dropping the stationarity requirement of increments for Lévy processes, we get the class of additive processes. The distributions of Lévy and additive processes at any time are infinitely divisible, that is, they have the nth roots in the convolution sense for any n. When a time is fixed, the class of Lévy processes is in one-to-one correspondence with the class of infinitely divisible distributions. Additive processes are described by systems of infinitely divisible distributions.

This book is intended to provide comprehensive basic knowledge of Lévy processes, additive processes, and infinitely divisible distributions with detailed proofs and, at the same time, to serve as an introduction to stochastic processes. As we deal with the simplest stochastic processes, we do not assume any knowledge of stochastic processes with a continuous parameter. Prerequisites for this book are of the level of the textbook of Billingsley [**34**] or that of Chung [**80**].

Making an additional assumption of selfsimilarity or some extensions of it on Lévy or additive processes, we get certain important processes. Such are stable processes, semi-stable processes, and selfsimilar additive processes. We give them systematic study. Correspondingly, stable, semi-stable, and selfdecomposable distributions are treated. On the other hand, the class of Lévy processes contains processes quite different from selfsimilar, and intriguing time evolution in distributional properties appears.

There are ten chapters in this book. They can be divided into three parts. Chapters 1 and 2 constitute the basic part. Essential examples and a major tool for the analysis are given in Chapter 1. The tool is to consider Fourier transforms of probability measures, called characteristic functions. Then, in Chapter 2, characterization of all infinitely divisible distributions is given. They give description of all Lévy processes and also of all additive processes. Chapters 3, 4, and 5 are the second part. They develop fundamental results on which subsequent chapters rely. Chapter 3 introduces selfsimilarity and other structures. Chapter 4 deals with decomposition of sample functions into jumps and continuous motions. Chapter 5 is on distributional properties. The third part ranges from Chapter 6 to Chapter 10. They are nearly independent of each other and treat major topics on Lévy processes such as subordination and density transformation, recurrence and transience, potential theory, Wiener–Hopf factorizations, and unimodality and multimodality.

We do not touch extensions of Lévy processes and infinitely divisible distributions connected with Lie groups, hypergroups, and generalized convolutions. There are many applications of Lévy processes to stochastic integrals, branching processes, and measure-valued processes, but they are not included in this book. Risk theory, queueing theory, and stochastic finance are active fields where Lévy processes often appear.

The original version of this book is *Kahou katei* written in Japanese, published by Kinokuniya at the end of 1990. The book is enlarged and material is rewritten. Many recent advances are included and a new chapter on potential theory is added. Exercises are now given to each chapter and their solutions are at the end of the volume.

For many years I have been happy in collaborating with Makoto Yamazato and Toshiro Watanabe. I was encouraged by Takeyuki Hida and Hiroshi Kunita to write the original Japanese book and the present book. Frank Knight and Toshiro Watanabe read through the manuscript and gave me numerous suggestions for correction of errors and improvement of presentation. Kazuyuki Inoue, Mamoru Kanda, Makoto Maejima, Yumiko Sato, Masaaki Tsuchiya, and Makoto Yamazato pointed out many inaccuracies to be eliminated. Part of the book was presented in lectures at the University of Zurich [**446**] as arranged by Masao Nagasawa. The preparation of this book was made in AMSLaTeX; Shinta Sato assisted me with the computer. My heartfelt thanks go to all of them.

Ken-iti Sato
Nagoya, 1999

Remarks on notation

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are, respectively, the collections of all positive integers, all integers, all rational numbers, all real numbers, and all complex numbers.

$\mathbb{Z}_+$, $\mathbb{Q}_+$, and $\mathbb{R}_+$ are the collections of nonnegative elements of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively.

For $x \in \mathbb{R}$, positive means $x > 0$; negative means $x < 0$. For a sequence $\{x_n\}$, increasing means $x_n \leq x_{n+1}$ for all n; decreasing means $x_n \geq x_{n+1}$ for all n. Similarly, for a real function f, increasing means $f(s) \leq f(t)$ for $s < t$, and decreasing means $f(s) \geq f(t)$ for $s < t$. When the equality is not allowed, we say strictly increasing or strictly decreasing.

$\mathbb{R}^d$ is the d-dimensional Euclidean space. Its elements $x = (x_j)_{j=1,\ldots,d}$, $y = (y_j)_{j=1,\ldots,d}$ are column vectors with d real components. The inner product is $\langle x, y\rangle = \sum_{j=1}^d x_j y_j$; the norm is $|x| = (\sum_{j=1}^d {x_j}^2)^{1/2}$. The word d-variate is used in the same meaning as d-dimensional.

For sets A and B, $A \subset B$ means that all elements of A belong to B. For $A, B \subset \mathbb{R}^d$, $z \in \mathbb{R}^d$, and $c \in \mathbb{R}$, $A+z = \{x+z\colon x \in A\}$, $A-z = \{x-z\colon x \in A\}$, $A+B = \{x+y\colon x \in A,\ y \in B\}$, $A-B = \{x-y\colon x \in A,\ y \in B\}$, $cA = \{cx\colon x \in A\}$, $-A = \{-x\colon x \in A\}$, $A \setminus B = \{x\colon x \in A \text{ and } x \notin B\}$, $A^c = \mathbb{R}^d \setminus A$, and $\mathrm{dis}(z, A) = \inf_{x\in A} |z-x|$. $\overline{A}$ is the closure of A.

$\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra of $\mathbb{R}^d$. For any $B \in \mathcal{B}(\mathbb{R}^d)$, $\mathcal{B}(B)$ is the σ-algebra of Borel sets included in B. $\mathcal{B}(B)$ is also written as $\mathcal{B}_B$.

Leb(B) is the Lebesgue measure of a set B. Leb$(\mathrm{d}x)$ is written $\mathrm{d}x$.

$\int g(x,y)\mathrm{d}_x F(x,y)$ is the Stieltjes integral with respect to x for fixed y.

The symbol δ_a represents the probability measure concentrated at a.

$[\mu]_B$ is the restriction of a measure μ to a set B.

The expression $\mu_1 * \mu_2$ represents the convolution of finite measures μ_1 and μ_2; $\mu^n = \mu^{n*}$ is the n-fold convolution of μ. When $n = 0$, μ^n is understood to be δ_0.

Sometimes $\mu(B)$ is written as μB. Thus $\mu(a,b] = \mu((a,b])$.

A non-zero measure means a measure not identically zero.

$1_B(x)$ is the indicator function of a set B, that is, $1_B(x) = 1$ for $x \in B$ and 0 for $x \notin B$.

$a \wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$.

The expression $\operatorname{sgn} x$ represents the sign function; $\operatorname{sgn} x = 1,\ 0,\ -1$ according as $x > 0,\ = 0,\ < 0$, respectively.

$P[A]$ is the probability of an event A. Sometimes $P[A]$ is written as PA.

$E[X]$ is the expectation of a random variable X. $E[X; A] = E[X 1_A]$. Sometimes $E[X]$ is written as EX.

$\operatorname{Var} X$ is the variance of a real random variable X.

$X \stackrel{\mathrm{d}}{=} Y$ means that X and Y are identically distributed. See p. 3 for the meaning of $\{X_t\} \stackrel{\mathrm{d}}{=} \{Y_t\}$.

P_X is the distribution of X.

The abbreviation a. s. denotes almost surely, that is, with probability 1. The abbreviation a. e. denotes almost everywhere, or almost every, with respect to the Lebesgue measure. Similarly, μ-a. e. denotes almost everywhere, or almost every, with respect to a measure μ.

$D([0,\infty), \mathbb{R}^d)$ is the collection of all functions $\xi(t)$ from $[0,\infty)$ to $\mathbb{R}^d$ such that $\xi(t)$ is right-continuous, $\xi(t+) = \lim_{h\downarrow 0} \xi(t+h) = \xi(t)$ for $t \geq 0$, and $\xi(t)$ has left limits $\xi(t-) = \lim_{h\downarrow 0} \xi(t-h) \in \mathbb{R}^d$ for $t > 0$.

I is the identity matrix. A' is the transpose of a matrix A. For an $n \times m$ real matrix A, $\|A\|$ is the operator norm of A as a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, that is, $\|A\| = \sup_{|x|\leq 1} |Ax|$. (However, the prime is sometimes used not in this way. For example, together with a stochastic process $\{X_t\}$ taking values in $\mathbb{R}^d$, we use $\{X'_t\}$ for another stochastic process taking values in $\mathbb{R}^d$; X'_t is not the transpose of X_t.)

Sometimes a subscript is written larger in parentheses, such as $X_t(\omega) = X(t,\omega)$, $X_t = X(t)$, $S_n = S(n)$, $T_x = T(x)$, $x_n = x(n)$, and $t_k = t(k)$.

The integral of a vector-valued function or the expectation of a random variable on $\mathbb{R}^d$ is a vector with componentwise integrals or expectations.

$\# A$ is the number of elements of a set A.

The expression $f(t) \sim g(t)$ means that $f(t)/g(t)$ tends to 1.

The symbol $\square$ denotes the end of a proof.