1

Statics and dynamics: some elementary concepts

Dynamics is the study of the movement through time of variables such as heartbeat, temperature, species population, voltage, production, employment, prices and so forth.

This is often achieved by means of equations linking the values of variables at different, uniformly spaced instants of time, i.e., **difference equations**, or by systems relating the values of variables to their time derivatives, i.e., **ordinary differential equations**. Dynamical phenomena can also be investigated by other types of mathematical representations, such as partial differential equations, lattice maps or cellular automata. In this book, however, we shall concentrate on the study of systems of difference and differential equations and their dynamical behaviour.

In the following chapters we shall occasionally use models drawn from economics to illustrate the main concepts and methods. However, in general, the mathematical properties of equations will be discussed independently of their applications.

1.1 A static problem

To provide a first, broad idea of the problems posed by dynamic *vis-à-vis* static analysis, we shall now introduce an elementary model that could be labelled as 'supply-demand-price interaction in a single market'. Our model considers the quantities supplied and demanded of a single good, defined as functions of a single variable, its price, p. In economic parlance, this would be called partial analysis since the effect of prices and quantities determined in the markets of all other goods is neglected. It is assumed that the demand function $D(p)$ is decreasing in p (the lower the price, the greater the amount that people wish to buy), while the supply function $S(p)$ is increasing in p (the higher the price, the greater the amount that people wish to supply).

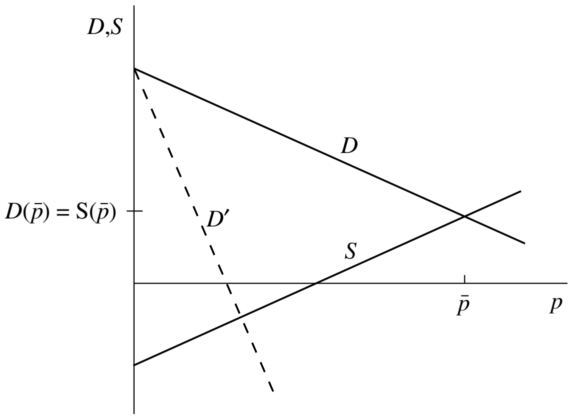

Fig. 1.1 The static partial equilibrium model

For example, in the simpler, linear case, we have:

$$\begin{aligned} D(p) &= a - bp \\ S(p) &= -m + sp \end{aligned} \tag{1.1}$$

and a, b, m and s are positive constants. Only nonnegative values of these variables are economically meaningful, thus we only consider $D, S, p \geq 0$. The **economic equilibrium condition** requires that the market of the good clears, that is demand equals supply, namely:

$$D(p) = S(p) \tag{1.2}$$

or

$$a - bp = -m + sp.$$

STATIC SOLUTION Mathematically, the solution to our problem is the value of the variable that solves (1.2) (in this particular case, a linear equation). Solving (1.2) for p we find:

$$\bar{p} = \frac{a+m}{b+s}$$

where $\bar{p}$ is usually called the **equilibrium price** (see figure 1.1).[1] We call the problem *static* since no reference is made to time or, if you prefer,

[1]The demand curve D' in figure 1.1 is provided to make the point that, with no further constraints on parameter values, the equilibrium price could imply negative equilibrium quantities of supply and demand. To eliminate this possibility we further assume that $0 < m/s \leq a/b$, as is the case for the demand curve D.

everything happens at the same time. Notice that, even though the static model allows us to find the equilibrium price of the good, it tells us nothing about what happens if the actual price is different from its equilibrium value.

1.2 A discrete-time dynamic problem

The introduction of dynamics into the model requires that we replace the equilibrium condition (1.2) with some hypothesis concerning the behaviour of the system off-equilibrium, i.e., when demand and supply are not equal. For this purpose, we assume the most obvious mechanism of price adjustment: over a certain interval of time, the price increases or decreases in proportion to the excess of demand over supply, $(D - S)$ (for short, **excess demand**). Of course, excess demand can be a positive or a negative quantity. Unless the adjustment is assumed to be instantaneous, prices must now be dated and p_n denotes the price of the good at time n, time being measured at equal intervals of length h. Formally, we have

$$p_{n+h} = p_n + h\theta[D(p_n) - S(p_n)]. \tag{1.3}$$

Since h is the period of time over which the adjustment takes place, θ can be taken as a measure of the speed of price response to excess demand. For simplicity, let us choose $h = 1$, $\theta = 1$. Then we have, making use of the demand and supply functions (1.1),

$$p_{n+1} = a + m + (1 - b - s)p_n. \tag{1.4}$$

In general, a solution of (1.4) is *a function of time* $p(n)$ (with n taking discrete, integer values) that satisfies (1.4).[2]

DYNAMIC SOLUTION To obtain the full dynamic solution of (1.4), we begin by setting $\alpha = a + m$, $\beta = (1 - b - s)$ to obtain

$$p_{n+1} = \alpha + \beta p_n. \tag{1.5}$$

To solve (1.5), we first set it in a canonical form, with all time-referenced terms of the variable on the left hand side (LHS), and all constants on the right hand side (RHS), thus:

$$p_{n+1} - \beta p_n = \alpha. \tag{1.6}$$

Then we proceed in steps as follows.

[2] We use the forms p_n and $p(n)$ interchangeably, choosing the latter whenever we prefer to emphasise that p is a function of n.

STEP 1 We solve the **homogeneous equation**, which is formed by setting the RHS of (1.6) equal to 0, namely:

$$p_{n+1} - \beta p_n = 0. \tag{1.7}$$

It is easy to see that a function of time $p(n)$ satisfying (1.7) is $p(n) = C\beta^n$, with C an arbitrary constant. Indeed, substituting in (1.7), we have

$$C\beta^{n+1} - \beta C\beta^n = C\beta^{n+1} - C\beta^{n+1} = 0.$$

STEP 2 We find a **particular solution** of (1.6), assuming that it has a form similar to the RHS in the general form. Since the latter is a constant, set $p(n) = k$, k a constant, and substitute it into (1.6), obtaining

$$k - \beta k = \alpha$$

so that

$$k = \frac{\alpha}{1-\beta} = \frac{a+m}{b+s} = \bar{p} \qquad \text{again!}$$

It follows that the $p(n) = \bar{p}$ is a solution to (1.6) and the constant (or stationary) solution of the dynamic problem is simply the solution of the static problem of section 1.1.

STEP 3 Since (1.6) is linear, the sum of the homogeneous and the particular solution is again a solution,[3] called the **general solution**. This can be written as

$$p(n) = \bar{p} + C\beta^n. \tag{1.8}$$

The arbitrary constant C can now be expressed in terms of the initial condition. Putting $p(0) \equiv p_0$, and solving (1.8) for C we have

$$p_0 = \bar{p} + C\beta^0 = \bar{p} + C$$

whence $C = p_0 - \bar{p}$, that is, the difference between the initial and equilibrium values of p. The general solution can now be re-written as

$$p(n) = \bar{p} + (p_0 - \bar{p})\beta^n. \tag{1.9}$$

Letting n take integer values $1, 2, \ldots$, from (1.9) we can generate a sequence of values of p, a 'history' of that variable (and consequently, a history of quantities demanded and supplied at the various prices), once its value at any arbitrary instant of time is given. Notice that, since the function $p_{n+1} =$

[3]This is called the *superposition principle* and is discussed in detail in chapter 2 section 2.1.

$f(p_n)$ is **invertible**, i.e., the function f^{-1} is well defined, $p_{n-1} = f^{-1}(p_n)$ also describes the past history of p.

The value of p at each instant of time is equal to the sum of the equilibrium value (the solution to the static problem which is also the particular, stationary solution) and the initial disequilibrium $(p_0 - \bar{p})$, amplified or dampened by a factor β^n. There are therefore two basic cases:

(i) $|\beta| > 1$. Any nonzero deviation from equilibrium is amplified in time, the equilibrium solution is unstable and as $n \to +\infty$, p_n asymptotically tends to $+\infty$ or $-\infty$.
(ii) $|\beta| < 1$. Any nonzero deviation is asymptotically reduced to zero, $p_n \to \bar{p}$ as $n \to +\infty$ and the equilibrium solution is consequently stable.

First-order, discrete-time equations (where the order is determined as the difference between the extreme time indices) can also have fluctuating behaviour, called **improper oscillations**,[4] owing to the fact that if $\beta < 0$, β^n will be positive or negative according to whether n is even or odd. Thus the sign of the adjusting component of the solution, the second term of the RHS of (1.9), oscillates accordingly. Improper oscillations are dampened if $\beta > -1$ and explosive if $\beta < -1$.

In figure 1.2 we have two representations of the motion of p through time. In figure 1.2(a) we have a line defined by the solution equation (1.5), and the bisector passing through the origin which satisfies the equation $p_{n+1} = p_n$. The intersection of the two lines corresponds to the constant, equilibrium solution. To describe the off-equilibrium dynamics of p, we start on the abscissa from an initial value $p_0 \neq \bar{p}$. To find p_1, we move vertically to the solution line and sidewise horizontally to the ordinate. To find p_2, we first reflect the value of p_1 by moving horizontally to the bisector and then vertically to the abscissa. From the point p_1, we repeat the procedure proposed for p_0 (up to the solution line, left to the ordinate), and so on and so forth. The procedure can be simplified by omitting the intermediate step and simply moving up to the solution line and sidewise to the bisector, up again, and so on, as indicated in figure 1.2(a). It is obvious that for $|\beta| < 1$, at each iteration of the procedure the initial deviation from equilibrium is diminished again, see figure 1.2(b). For example, if $\beta = 0.7$, we have $\beta^2 = 0.49,\ \beta^3 = 0.34, \ldots, \beta^{10} \approx 0.03, \ldots)$ and the equilibrium solution is approached asymptotically.

The reader will notice that stability of the system and the possibility

[4] The term *improper* refers to the fact that in this case oscillations of variables have a ‘kinky’ form that does not properly describe the smoother ups and downs of real variables. We discuss *proper* oscillations in chapter 3.

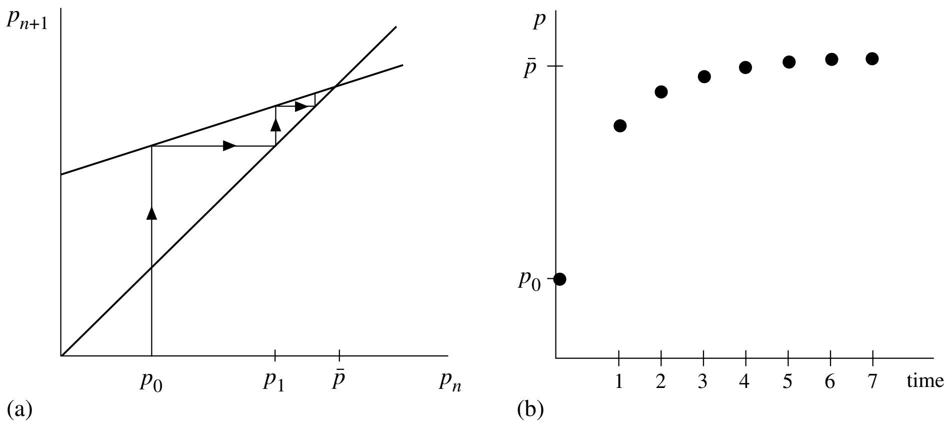

Fig. 1.2 Convergence to $\bar{p}$ in the discrete-time partial equilibrium model

of oscillatory behaviour depends entirely on β and therefore on the two parameters b and s, these latter denoting respectively the slopes of the demand and supply curves. The other two parameters of the system, a and m, determine α and consequently they affect only the equilibrium value $\bar{p}$. We can therefore completely describe the dynamic characteristics of the solution (1.9) over the parameter space (b, s). The boundary between stable and unstable behaviour is given by $|\beta| = 1$, and convergence to equilibrium is guaranteed for

$$-1 < \beta < 1$$
$$2 > b + s > 0.$$

The assumptions on the demand and supply functions imply that $b, s > 0$. Therefore, the stability condition is $(b + s) < 2$, the stability boundary is the line $(b + s) = 2$, as represented in figure 1.3. Next, we define the curve $\beta = 1 - (b + s) = 0$, separating the zone of monotonic behaviour from that of improper oscillations, which is also represented in figure 1.3. Three zones are labelled according to the different types of dynamic behaviour, namely: convergent and monotonic; convergent and oscillatory; divergent and oscillatory. Since $b, s > 0$, we never have the case $\beta > 1$, corresponding to divergent, nonoscillatory behaviour.

If $|\beta| > 1$ any initial difference $(p_0 - \bar{p})$ is amplified at each step. In this model, we can have $|\beta| > 1$ if and only if $\beta < -1$. Instability, then, is due to **overshooting**. Any time the actual price is, say, too low and there is positive excess demand, the adjustment mechanism generates a change in the price in the 'right' direction (the price rises) but the change is too large.

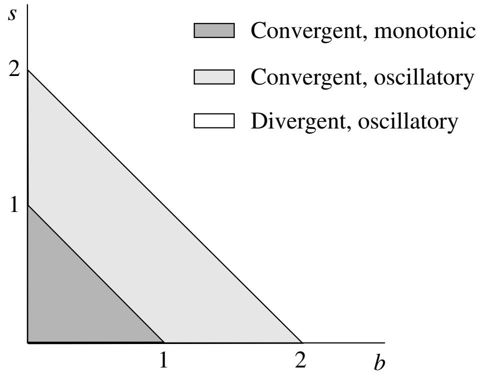

Fig. 1.3 Parameter space for the discrete-time partial equilibrium model

After the correction, the new price is too high (negative excess demand) and the discrepancy from equilibrium is larger than before. A second adjustment follows, leading to another price that is far too low, and so on. We leave further study of this case to the exercises at the end of this chapter.

1.3 A continuous-time dynamic problem

We now discuss our simple dynamical model in a continuous-time setting. Let us consider, again, the price adjustment equation (1.3) (with $\theta = 1$, $h > 0$) and let us adopt the notation $p(n)$ so that

$$p(n+h) = p(n) + h\left(D[p(n)] - S[p(n)]\right).$$

Dividing this equation throughout by h, we obtain

$$\frac{p(n+h) - p(n)}{h} = D[p(n)] - S[p(n)]$$

whence, taking the limit of the LHS as $h \to 0$, and recalling the definition of a derivative, we can write

$$\frac{dp(n)}{dn} = D[p(n)] - S[p(n)].$$

Taking the interval h to zero is tantamount to postulating that time is a continuous variable. To signal that time is being modelled differently we substitute the time variable $n \in \mathbb{Z}$ with $t \in \mathbb{R}$ and denote the value of p at time t simply by p, using the extended form $p(t)$ when we want to emphasise that price is a function of time. We also make use of the efficient Newtonian notation $dx(t)/dt = \dot{x}$ to write the price adjustment mechanism as

$$\frac{dp}{dt} = \dot{p} = D(p) - S(p) = (a+m) - (b+s)p. \tag{1.10}$$

Equation (1.10) is an ordinary differential equation relating the values of the variable p at a given time t to its first derivative with respect to time at the same moment. It is **ordinary** because the solution $p(t)$ is a function of a single independent variable, time. Partial differential equations, whose solutions are functions of more than one independent variable, will not be treated in this book, and when we refer to differential equations we mean ordinary differential equations.

DYNAMIC SOLUTION The dynamic problem is once again that of finding a function of time $p(t)$ such that (1.10) is satisfied for an arbitrary initial condition $p(0) \equiv p_0$.

As in the discrete-time case, we begin by setting the equation in canonical form, with all terms involving the variable or its time derivatives on the LHS, and all constants or functions of time (if they exist) on the RHS, thus

$$\dot{p} + (b+s)p = a+m. \tag{1.11}$$

Then we proceed in steps as follows.

STEP 1 We solve the homogeneous equation, formed by setting the RHS of (1.11) equal to 0, and obtain

$$\dot{p} + (b+s)p = 0 \text{ or } \dot{p} = -(b+s)p. \tag{1.12}$$

If we now integrate (1.12) by separating variables, we have

$$\int \frac{dp}{p} = -(b+s)\int dt$$

whence

$$\ln p(t) = -(b+s)t + A$$

where A is an arbitrary integration constant. Taking now the antilogarithm of both sides and setting $e^A = C$, we obtain

$$p(t) = Ce^{-(b+s)t}.$$

STEP 2 We look for a particular solution to the nonhomogeneous equation (1.11). The RHS is a constant so we try $p = k$, k a constant and consequently $\dot{p} = 0$. Therefore, we have

$$\dot{p} = 0 = (a + m) - (b + s)k$$

whence

$$k = \frac{a + m}{b + s} = \bar{p}.$$

Once again the solution to the static problem turns out to be a special (stationary) solution to the corresponding dynamic problem.

STEP 3 Since (1.12) is linear, the general solution can be found by summing the particular solution and the solution to the homogeneous equation, thus

$$p(t) = \bar{p} + Ce^{-(b+s)t}.$$

Solving for C in terms of the initial condition, we find

$$p(0) \equiv p_0 = \bar{p} + C \text{ and } C = (p_0 - \bar{p}).$$

Finally, the complete solution to (1.10) in terms of time, parameters, initial and equilibrium values is

$$p(t) = \bar{p} + (p_0 - \bar{p})e^{-(b+s)t}. \qquad (1.13)$$

As in the discrete case, the solution (1.13) can be interpreted as the sum of the equilibrium value and the initial deviation of the price variable from equilibrium, amplified or dampened by the term $e^{-(b+s)t}$. Notice that in the continuous-time case, a solution to a differential equation $\dot{p} = f(p)$ always determines both the future and the past history of the variable p, independently of whether the function f is invertible or not. In general, we can have two main cases, namely:

(i) $(b + s) > 0$ Deviations from equilibrium tend asymptotically to zero as $t \to +\infty$.
(ii) $(b + s) < 0$ Deviations become indefinitely large as $t \to +\infty$ (or, equivalently, deviations tend to zero as $t \to -\infty$).

Given the assumptions on the demand and supply functions, and therefore on b and s, the explosive case is excluded for this model. If the initial price is below its equilibrium value, the adjustment process ensures that the price increases towards it, if the initial price is above equilibrium, the price declines to it. (There can be no overshooting in the continuous-time case.) In a manner analogous to the procedure for difference equations, the equilibria

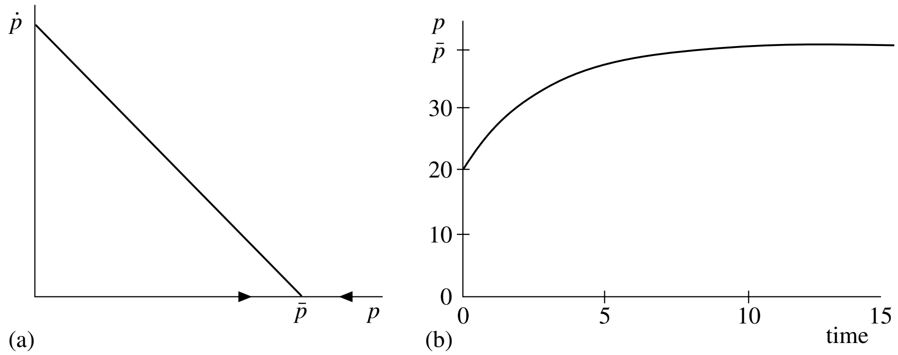

Fig. 1.4 The continuous-time partial equilibrium model

of differential equations can be determined graphically in the plane $(p, \dot{p})$ as suggested in figure 1.4(a). Equilibria are found at points of intersection of the line defined by (1.10) and the abscissa, where $\dot{p} = 0$. Convergence to equilibrium from an initial value different from the equilibrium value is shown in figure 1.4(b).

Is convergence likely for more general economic models of price adjustment, where other goods and income as well as substitution effects are taken into consideration? A comprehensive discussion of these and other related microeconomic issues is out of the question in this book. However, in the appendixes to chapter 3, which are devoted to a more systematic study of stability in economic models, we shall take up again the question of convergence to or divergence from economic equilibrium.

We would like to emphasise once again the difference between the discrete-time and the continuous-time formulation of a seemingly identical problem, represented by the two equations

$$p_{n+1} - p_n = (a + m) - (b + s)p_n \tag{1.4}$$

$$\dot{p} = (a + m) - (b + s)p. \tag{1.10}$$

Whereas in the latter case $(b+s) > 0$ is a sufficient (and necessary) condition for convergence to equilibrium, stability of (1.4) requires that $0 < (b+s) < 2$, a tighter condition.

This simple fact should make the reader aware that a naive translation of a model from discrete to continuous time or vice versa may have unsuspected consequences for the dynamical behaviour of the solutions.