AN INTRODUCTION TO FINANCIAL OPTION VALUATION

Mathematics, Stochastics and Computation

This is a lively textbook providing a solid introduction to financial option valuation for undergraduate students armed with only a working knowledge of first year calculus. Written as a series of short chapters, this self-contained treatment gives equal weight to applied mathematics, stochastics and computational algorithms, with no prior background in probability, statistics or numerical analysis required.

Detailed derivations of both the basic asset price model and the Black–Scholes equation are provided along with a presentation of appropriate computational techniques including binomial, finite differences and, in particular, variance reduction techniques for the Monte Carlo method.

Each chapter comes complete with accompanying stand-alone MATLAB code listing to illustrate a key idea. The author has made heavy use of figures and examples, and has included computations based on real stock market data. Solutions to exercises are made available at www.cambridge.org.

DES HIGHAM is a professor of mathematics at the University of Strathclyde. He has co-written two previous books, *MATLAB Guide* and *Learning LaTeX*. In 2005 he was awarded the Germund Dahlquist Prize by the Society for Industrial and Applied Mathematics for his research contributions to a broad range of problems in numerical analysis.

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To my family,
Catherine, Theo, Sophie and Lucas

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Preface

The aim of this book is to present a lively and palatable introduction to financial option valuation for undergraduate students in mathematics, statistics and related areas. Prerequisites have been kept to a minimum. The reader is assumed to have a basic competence in calculus up to the level reached by a typical first year mathematics programme. No background in probability, statistics or numerical analysis is required, although some previous exposure to material in these areas would undoubtedly make the text easier to assimilate on first reading.

The contents are presented in the form of short chapters, each of which could reasonably be covered in a one hour teaching session. The book grew out of a final year undergraduate class called *The Mathematics of Financial Derivatives* that I have taught, in collaboration with Professor Xuerong Mao, at the University of Strathclyde. The class is aimed at students taking honours degrees in Mathematics or Statistics, or joint honours degrees in various combinations of Mathematics, Statistics, Economics, Business, Accounting, Computer Science and Physics. In my view, such a class has two great selling points.

- From a student perspective, the topic is generally perceived as modern, sexy and likely to impress potential employers.
- From the perspective of a university teacher, the topic provides a focus for ideas from mathematical modelling, analysis, stochastics and numerical analysis.

There are many excellent books on option valuation. However, in preparing notes for a lecture course, I formed the opinion that there is a niche for a single, self-contained, introductory text that gives equal weight to

- applied mathematics,
- stochastics, and
- computational algorithms.

The classic applied mathematics view is provided by Wilmott, Howison and Dewynne's text (Wilmott *et al.*, 1995). My aim has been to write a book at a similar level with a less ambitious scope (only option valuation is considered), less emphasis on partial differential equations, and more attention paid to stochastic modelling and simulation.

Key features of this book are as follows.

(i) Detailed derivation and discussion of the basic lognormal asset price model.
(ii) Roughly equal weight given to binomial, finite difference and Monte Carlo methods. In particular, variance reduction techniques for Monte Carlo are treated in some detail.
(iii) Heavy use of computational examples and figures as a means of illustration.
(iv) Stand-alone MATLAB codes, with full listings and comprehensive descriptions, that implement the main algorithms. The core text can be read independently of the codes. Readers who are familiar with other programming languages or problem-solving environments should have little difficulty in translating these examples.

In a nutshell, this is the book that I wish had been available when I started to prepare lectures for the Strathclyde class.

When designing a text like this, an immediate issue is the level at which stochastic calculus is to be treated. One of the tenets of this book is that

> rigorous, measure-theoretic, stochastic analysis, although beautiful, is *hard* and it is unrealistic to ask an undergraduate class to pick up such material on the fly. Monte Carlo-style simulation, on the other hand, is a relatively *simple* concept, and well-chosen computational experiments provide an excellent way to back up heuristic arguments.

Hence, the approach here is to treat stochastic calculus on a nonrigorous level and give plenty of supporting computational examples. I rely heavily on the Central Limit Theorem as a basis for heuristic arguments. This involves a deliberate compromise – convergence in distribution must be swapped for a stronger type of convergence if these arguments are to be made rigorous – but I feel that erring on the side of accessibility is reasonable, given the aims of this text.

In fact, in deriving the Black–Scholes partial differential equation, I do not make explicit reference to Itô's Lemma. I decided that a heuristic derivation of Itô's Lemma in a general setting followed by a single application of the lemma in one simple case makes less pedagogical sense than a direct '*in situ*' heuristic treatment, a decision inspired by Almgren's expository article (Almgren, 2002). I hope that at least some undergraduate readers will be sufficiently motivated to follow up on the references and become exposed to the real thing.

You can get a feeling for the contents of the book by skimming through the outline bullet points that appear at the start of each chapter. Many of the later chapters can be read independently of each other, or, of course, omitted.

Exercises are given at the end of each chapter. It is my experience that active problem solving is the best learning tool, so I strongly encourage students to make use of them. I have used a starring system: one star for questions whose solution is relatively easy/short, rising to three stars for the hardest/longest questions. Brief solutions to the odd-numbered exercises are available from the book website given below. This leaves the even-numbered questions as a teaching resource. Certain questions are central to the text. I have tried to ensure that these come up in the odd-numbered list, in order to aid independent study.

A short, introductory treatment like this can only scratch the surface. Hence, each chapter concludes with a *Notes and references* section, which gives my own, necessarily biased, hints about important omissions. References can be followed up via the *References* section at the end of the book.

Scattered at the end of each chapter are a few quotes, designed to enlighten and entertain. Some of these reinforce the ideas in the text and others cast doubt on them. Mathematical option valuation is a strange business of sophisticated analysis based on simple models that have obvious flaws and perhaps do not merit such detailed scrutiny. When preparing lecture notes, I have found that authoritative, pithy quotes are a particularly powerful means to highlight some of this tension. I have an uneasy feeling that some Strathclyde students spent more time perusing the quotes than the main text, so I have aimed to make the quotes at least form a reasonable mini-summary of the contents. Most quotes relate directly to their chapter, but a few general ones have been dispersed throughout the book on the grounds that they were too good to leave out.

A website for this book has been created at www.maths.strath.ac.uk/~aas96106/option_book.html. It includes the following.

- The MATLAB codes listed in the book.
- Outline solutions to the odd-numbered exercises.
- Links to the websites mentioned in the book.
- Colour versions of some of the figures.
- A list of corrections.
- Some extra quotes that did not make it into the book.

I am grateful to several people who have influenced this book. **Nick Higham** cast a critical eye over an early draft and made many helpful suggestions. **Vicky Henderson** checked parts of the text and patiently answered a number of questions. **Petter Wiberg** gave me access to his MATLAB files for processing stock market data. **Xuerong Mao**, through animated discussions and research collaboration, has enriched my understanding of stochastics and its role in mathematical finance. Additionally, five anonymous reviewers provided unbiased feedback. In particular, one reviewer who was not in favour of the nonrigorous approach to stochastic analysis in this book was nevertheless generous enough to provide detailed comments that allowed me to improve the final product. Finally, three years'

worth of Strathclyde honours students have helped to shape my views on how to present this material to a wide audience.

MATLAB programs

I firmly believe that the best way to check your understanding of a computational algorithm is to examine, and interactively experiment with, a real program. For this reason, I have included a *Program of the Chapter* at the end of every chapter, followed by two programming exercises. Each program illustrates a key topic. They can be downloaded from the website previously mentioned.

The programs are written in MATLAB.[1] I chose this environment for a number of reasons.

- It offers excellent random number generation and graphical output facilities.
- It has powerful, built-in, high-level commands for matrix computations and statistics.
- It runs on a variety of platforms.
- It is widely available in mathematics and computer science departments and is often used as the basis for scientific computing or numerical analysis courses. Students may purchase individual copies at a modest price.

I wrote the programs with *accuracy* and *clarity* in mind, rather than efficiency or elegance. I have made quite heavy use of MATLAB's vectorization facilities, where possible working with arrays directly and eschewing unnecessary `for` loops. This tends to make the codes shorter, snappier and less daunting than alternatives that operate on individual array components. Meaningful comments have been inserted into the codes and a 'walkthrough' commentary is appended in each case. Those walkthroughs provide MATLAB information on a just-in-time basis. For a comprehensive guide to MATLAB, see (Higham and Higham, 2000).

I have not made use of any of the toolboxes that are available, at extra cost, to MATLAB users. This is because (a) the emphasis in the book is on understanding the underlying models and algorithms, not on the use of black-box packages, and (b) only a small percentage of MATLAB users will have access to toolboxes. However, those who wish to perform serious option valuation computations in MATLAB are advised to investigate the toolboxes, especially those for Finance, Statistics, Optimization and PDEs.

Readers with some experience of scientific computing in languages such as Java, C or FORTRAN should find it relatively easy to understand the codes. Those with no computing background may need to put in more effort, but should find the process rewarding.

[1] MATLAB is a registered trademark of The MathWorks, Inc.

MATLAB is a commercial software product produced by The Mathworks, whose homepage is at www.mathworks.com/.

Let me re-emphasize that these programs are entirely stand-alone; the book can be read without reference to them. However, I believe that they form a major element – if you understand the programs, you understand a big chunk of the material in this book.

Disclaimer of warranty

We make no warranties, express or implied, that the programs contained in this volume are free of error, or are consistent with any particular standard of merchantability, or that they will meet your requirements for any particular application. They should not be relied on for solving a problem whose incorrect solution could result in injury to a person or loss of property. If you do use the programs in such a manner, it is at your own risk. The author and publisher disclaim all liability for direct or consequential damages resulting from your use of the programs.