1

Symmetries and conservation laws

1.1 Symmetries

Symmetry is one of the most fruitful principles in science. In fact, nowadays acquaintance with the structures that describe symmetries is an indispensable prerequisite for anybody who wants to become familiar with the deep and beautiful concepts of modern physics. Besides their aesthetic appeal, the virtue of symmetries is that they enable us to make precise quantitative predictions about various physical systems, which can be compared with experiment.

The concept of symmetry is extremely general, and correspondingly the precise meaning of the term depends to a large extent on the setting. For the purposes of the present book, we reserve the word symmetry for mappings of the physical states of a system which leave the dynamics invariant. These operations must be either invertible or infinitesimal analogues of invertible mappings. What is meant by the terms ‘invariant’ and ‘dynamics’ still depends on the framework which is used to study the system. In the rest of this chapter several such frameworks will be presented, and each time the specific meaning of ‘symmetry’ will be explained. In all these examples the structure of Lie algebras emerges in the description of the symmetries. In chapter 2 and chapter 3 we will analyze specific examples of such Lie algebra symmetries which are still quite simple but nevertheless already display much of the general structure. Only afterwards, from chapter 4 on, we investigate the general theory of Lie algebras and related structures. Those readers who already have enough background or who are exclusively interested in the mathematical aspects of the description of symmetries may proceed directly to chapter 4.

Already before the restriction to a particular framework some fundamental observations can be made. Namely, first, the composition of mappings is always associative; second, there is the identity map which triv-
1 Symmetries and conservation laws

Initially respects the dynamics; and third, the mappings are invertible by assumption. This way the mathematical structure of a group of symmetries arises in a natural manner.

1.2 Continuous parameters and local one-parameter groups

In many examples, such as translations or rotations, the group elements that describe symmetries of physical systems can be labelled by a continuous parameter, like the angle of a rotation or the translation vector, on which the group elements depend in a differentiable manner. In this situation, one can also study ‘infinitesimal’ symmetries. Such an infinitesimal symmetry is also called a generator of a symmetry, because often there are methods to recover group elements from these infinitesimal symmetries. A basic ingredient of a group is the operation of multiplication, which associates to any two group elements a third one. This operation has an infinitesimal counterpart as well, which associates to two infinitesimal symmetries a third infinitesimal symmetry. However, this operation has no longer the properties of a group multiplication; rather, it will lead us to introduce the mathematical structure of an algebra.

To characterize an infinitesimal symmetry, it is actually sufficient to know how the group elements look in some small neighborhood of the identity element of the group. Suppose we are given a continuous parametrization \( t \mapsto \gamma(t) \) of group elements, where the parameter \( t \) is a real number, and assume for simplicity that \( \gamma(0) \) is the unit element of the group. This parametrization should be such that the product of two group elements \( \gamma(t_1) \) and \( \gamma(t_2) \) satisfies \( \gamma(t_1)\gamma(t_2) = \gamma(t_1 + t_2) \). An infinitesimal symmetry will be described by the derivative of the function \( \gamma(t) \) at \( t = 0 \). To construct such a generator, it is not necessary to know the function \( \gamma(t) \) for all values of \( t \), and it is not even necessary that such a function is defined for all values of \( t \). Rather, to obtain an infinitesimal symmetry it is sufficient to have a function \( \gamma \) obeying \( \gamma(t_1)\gamma(t_2) = \gamma(t_1 + t_2) \) that is only defined on some small interval around 0. Such a function is sometimes called a local one-parameter group; the qualification ‘local’ refers to the fact that the function is only defined near zero.

Already this simple consideration indicates that there may exist symmetries of physical systems which cannot be described by groups. We will see shortly that, just like symmetry groups, such symmetries can give rise to conserved quantities and therefore contain valuable information.

1.3 Classical mechanics: Lagrangian description

In the Lagrangian description of classical mechanics, a system is characterized by the Lagrangian \( L \), which is a function of positions \( q_i \) and of the
1.4 Conservation laws

tangent vectors to position space, i.e. the velocities $\dot{q}_i$. To describe the characteristic features of this framework, it suffices to restrict attention to the case of a single position $q$ and velocity $\dot{q}$. The solutions to the equations of motion are precisely the extrema of the action, i.e. of the time integral $S = \int L(t) \, dt$ of the Lagrangian (subject to the requirement that the boundary conditions are kept fixed). Hence the equations of motion are given by the \textit{Euler–Lagrange equations}

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$  \hspace{1cm} (1.1)

Now suppose that for any real number $s$ in some small interval around $0$, $s \in (-\varepsilon, \varepsilon)$, there is a mapping $h_s$ of the position variable according to

$$h_s : \quad q \mapsto h_s(q),$$  \hspace{1cm} (1.2)

which is accompanied by an induced linear mapping $\dot{h}_s$ of the velocity $\dot{q}$, i.e.

$$\dot{h}_s : \quad \dot{q} \mapsto \dot{h}_s(\dot{q}) = \frac{\partial h_s(q)}{\partial q} \dot{q}.$$  \hspace{1cm} (1.3)

The Lagrangian $L$ is said to be \textit{invariant} under the transformation (1.2) and (1.3) if for any fixed $s$ there exists a function $F_s(q, \dot{q}, t)$ that can also depend on time, such that

$$L(h_s(q), \dot{h}_s(\dot{q})) = L(q, \dot{q}) + \frac{d}{dt} F_s.$$  \hspace{1cm} (1.4)

Note that we allowed for a total time derivative of an arbitrary function $F_s$; such an additional term is allowed because it does not change the equations of motion. Upon integration over some time interval $[t_1, t_2]$, the total derivative gives rise to a ‘surface term’ $F_s(q(t_2), \dot{q}(t_2), t_2) - F_s(q(t_1), \dot{q}(t_1), t_1)$ which only receives contributions from the boundary of the time interval. Surface terms play an important role in several applications. For instance, Lagrangians can be invariant under a \textit{supersymmetry} (see section 20.9) only up to such terms.

1.4 Conservation laws

Any invariance of the type (1.4) gives rise to a conservation law. This statement, known as \textit{Noether’s theorem}, can be derived as follows. A mapping

$$\phi : \quad t \mapsto q = \phi(t)$$  \hspace{1cm} (1.5)

from a time interval to the position space is called a \textit{path}. We can now use any symmetry $h_s$ to map such a path $\phi$ to a new path $\phi_s$:

$$\phi_s = h_s \circ \phi : \quad t \mapsto q = h_s(\phi(t)).$$  \hspace{1cm} (1.6)
1 Symmetries and conservation laws

This way we obtain a whole family of paths, one path for every value of \( s \). The invariance (1.4) tells us that the value of the action of the path \( \Phi_s \) is independent of \( s \). In particular, if the path (1.5) is a solution of the equations of motion (1.1), the action is minimal on the path \( \Phi \) and, by the invariance, it is minimal on any other path \( \Phi_s \) as well. In other words, any path \( \Phi_s \) is a solution of the equations of motion.

To investigate this whole family of paths, it is convenient to treat the variable \( t \) parametrizing the individual path and the variable \( s \) that labels the different paths on an equal footing. Thus we introduce a function \( \Phi \) of two variables,

\[ \Phi(s, t) := h_s \circ \phi(t), \]

i.e. for fixed value of \( s \) one just obtains the path \( \phi_s \). With the help of the invariance (1.4) it follows that \( \Phi \) satisfies

\[ 0 = \frac{\partial}{\partial s} \left( \mathcal{L}(\Phi, \frac{\partial \Phi}{\partial t}) - \frac{d}{dt} F_s \right) = \frac{\partial L}{\partial \Phi} \frac{\partial \Phi}{\partial s} + \frac{\partial L}{\partial \Phi} \frac{\partial \Phi}{\partial t} \frac{\partial s}{\partial t} - \frac{d}{dt} F_s. \]

Using the equations of motion (1.1), this can be rewritten as

\[ 0 = \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \frac{\partial \Phi}{\partial s} + \frac{\partial L}{\partial \Phi} \frac{\partial \Phi}{\partial t} \frac{\partial s}{\partial t} - \frac{d}{dt} F_s = \frac{d}{dt} \left( \frac{\partial L}{\partial \Phi} \frac{\partial \Phi}{\partial s} - \frac{\partial F_s}{\partial s} \right). \]

This shows that the quantity

\[ Q := \frac{\partial L}{\partial \Phi} \frac{\partial \Phi}{\partial s} - \frac{\partial F_s}{\partial s} \]

does not depend on time, i.e. is a conserved quantity (also called a conserved charge or Noether charge). Note that in the derivation of this result we used the equations of motion; correspondingly the statement that the quantity \( Q \) does not depend on time means that it is constant in time for any solution to the equations of motion.

As an illustration, let us employ Noether’s theorem to derive the conservation of momentum for the motion of a point particle in \( \mathbb{R}^3 \). Suppose that the Lagrangian \( L \) is of the form \( L = \frac{1}{2} m \ddot{x}^2 - V(x) \), with \( x \in \mathbb{R}^3 \), and that the potential \( V(x) \) is invariant under a shift \( x \mapsto x + s \vec{a} \), i.e. \( V(x + s \vec{a}) = V(x) \) for all \( s \). Then also \( L \) is invariant under the transformation

\[ h_s : \ x \mapsto x + s \vec{a}, \]

while the mapping \( h_s \) is the identity map and \( F_s \equiv 0 \) for all \( s \). By computing

\[ \frac{\partial \Phi}{\partial s} = \frac{d}{ds} \frac{\partial \Phi}{\partial s} = \vec{a} \]
1.4 Conservation laws

we find that the conserved quantity (1.10) is

\[ Q_a = m \vec{x} \cdot \vec{a}. \]  

That is, the conserved charge associated to the invariance under the translation \( \vec{x} \mapsto \vec{x} + s\vec{a} \) is the momentum in \( \vec{a} \)-direction.

Noether’s theorem shows that any differentiable family of symmetry transformations yields a conservation law and a conserved charge \( Q \). We have just seen that, if the system is invariant under translations, one obtains conservation of momentum; similarly (see exercise 1.1), invariance under rotations leads to conservation of angular momentum. The theorem can also be easily generalized from mechanics to classical field theories. However, we warn the reader that the converse of Noether’s theorem does not hold: There can be conserved quantities, such as the so-called topological charges, which are not Noether charges. For example, it can happen that the configuration space, i.e. the space spanned by the (generalized) coordinates of the system, is not connected. As the motion is continuous and hence can never connect configurations in different components, it takes place in a single connected component. Thus when we attach a label, say an integer, to each component, we can interpret this label as the value of a conserved quantity.

To choose a sensible labelling corresponding to a topological charge can often be a complicated task. Typically the counting of connected components of the configuration space is most conveniently described by relating it to topological properties of other objects. An example for this is provided by four-dimensional Yang-Mills theories, where a suitable labelling of the connected components is given by the instanton number or Pontryagin index of the gauge field configuration (see e.g. chapter 9.4.1 of [Nakahara 1990]).

Noether’s theorem can in particular be applied to any continuous, differentiable group of symmetries. However, in the derivation of the theorem only a local one-parameter family of symmetries was used; thus the conserved quantities are actually associated not so much to the group, but rather to the infinitesimal version of the symmetry transformations; these form an algebra rather than a group. Algebras and groups are related by a certain exponential mapping (which will be studied in chapter 9). However, not any local family of symmetries can be integrated up to a group by exponentiation. In other words, there can be symmetries which can only be described by algebras, and these symmetries imply additional conservation laws. In fact, this situation is encountered rather frequently in modern physics; examples are provided by ‘supersymmetry’ (see section 20.9) and by ‘quantum groups’ (see chapter 22). Another important situation where this phenomenon occurs is conformal symmetry in a two-dimensional space: While the dimension of the conformal
1 Symmetries and conservation laws

The group (over $\mathbb{R}$) is six, i.e. in particular finite, the conformal algebra in two dimensions is the so-called Virasoro algebra (see section 12.12) which is infinite-dimensional; thus only a tiny part of the symmetries can be regarded as the infinitesimal version of a group transformation. An additional virtue of infinitesimal symmetries is that they are described by a linear structure, whereas the structure of groups is in general inherently non-linear.

1.5 Classical mechanics: Hamiltonian description

In the Hamiltonian framework of classical mechanics, all information is encoded in the Hamiltonian, a function of the positions $q_i$ and momenta $p_i$ (and possibly of the time $t$). The space spanned by the positions and momenta is called the phase space, to be distinguished from the configuration space, i.e. the space spanned only by the positions, which is the space in which Lagrangian dynamics takes place. Considering again the case of a single position variable $q$, the equations of motion read

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}. \quad (1.14)$$

For dynamical systems which can be treated both in the Hamiltonian and in the Lagrangian description, the relation between the two is given by a Legendre transformation, $H(q,p) = p\dot{q} - L$, where $\dot{q}$ is implicitly defined by $p = \partial L / \partial \dot{q}$.

However, the relation between Hamiltonian and Lagrangian classical dynamical systems is not one-to-one. On one hand, not any phase space of a Hamiltonian system (i.e. in mathematical terms, a symplectic manifold with a Hamiltonian vector field) can be described as the cotangent bundle of some Lagrangian submanifold; this is e.g. definitely not possible if the symplectic manifold is compact. On the other hand, in many interesting examples of Lagrangian dynamics some canonical momenta vanish, so that it is not possible to express $q$ in terms of $p$ and $q$ as needed in the Legendre transform. A handle on this situation is provided by the theory of constrained systems; for details about constrained dynamics see e.g. [Henneaux and Teitelboim 1992] and [Sundermeyer 1982].

One of the fundamental tools employed in the Hamiltonian formulation of mechanics is the Poisson bracket. This maps any pair $f, g$ of differentiable functions of the dynamical variables $p$ and $q$ to a single function of $p$ and $q$ which is denoted by $\{f, g\}$ and is obtained as

$$\{f, g\} := \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p}. \quad (1.15)$$

It is easy to check (see exercise 1.2) that the Poisson bracket possesses the following three characteristic properties:
1.5 Classical mechanics: Hamiltonian description

First, it is bilinear, i.e. satisfies

\[ \{ \lambda f + \mu g, h \} = \lambda \{ f, h \} + \mu \{ g, h \} \]  
\[(1.16)\]

for all real numbers \( \lambda, \mu \) and an analogous formula for \( \{ f, \lambda g + \mu h \} \).

Second, it is antisymmetric:

\[ \{ f, g \} = -\{ g, f \} . \]  
\[(1.17)\]

And third, the so-called Jacobi identity

\[ \{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0 \]  
\[(1.18)\]

holds.

These three properties can be summarized by the statement that the Poisson bracket \(\{ \cdot, \cdot \}\) turns the space of functions of \( p \) and \( q \) into what is called a ‘Lie algebra’.

Let us now assume that the Hamiltonian \( H \) does not depend on time, \( \partial H / \partial t = 0 \). (Sometimes this is described by saying that \( H \) does not depend explicitly on time, since after evaluating \( H \) on some specific trajectory, say a solution to the equations of motion, \( H \) would depend implicitly on \( t \) through \( p(t) \) and \( q(t) \). However, the correct point of view is to regard \( H \) just as a function of the two indeterminates \( p \) and \( q \), so that this qualification is unnecessary.) The time derivative of any function \( f(p(t), q(t)) \) on a trajectory that is a solution to the equations of motions can then be written as the Poisson bracket with the Hamiltonian (see exercise 1.2): \[ \frac{\partial}{\partial t} f(p(t), q(t)) = \{ H, f \} . \]  
\[(1.19)\]

A function \( f \) describes a conserved quantity if for any solution to the equations of motion the time derivative on the left hand side vanishes, and hence if it ‘Poisson-commutes’ with the Hamiltonian,

\[ \{ H, f \} = 0 . \]  
\[(1.20)\]

Such functions are said to be in involution with the Hamiltonian. Using the Jacobi identity (1.18), it follows (see exercise 1.2) that, if \( f \) and \( g \) are conserved, then \( \{ f, g \} \) is a conserved quantity, too. In other words, the space of functions that describe conserved quantities closes under the Poisson bracket. In fact, it constitutes in itself a Lie algebra, a subalgebra of the space of all functions. Hence, to learn more on the conserved charges of a system described by classical Hamiltonian mechanics, it is necessary to investigate the structure of Lie algebras. The Jacobi identity which ensures that together with \( f \) and \( g \) also \( \{ f, g \} \) is conserved, is a crucial and non-trivial input required for having the structure of a Lie algebra.

The presence of conserved quantities puts severe constraints on the dynamics of a system. Any trajectory that fulfills the equations of motion
remains inside a subspace of the phase space in which each conserved quantity takes a fixed value. If there are, for a phase space of dimension $2n$, $n$ conserved quantities which all Poisson commute, this subspace is typically an $n$-dimensional torus, if it is compact.

In the Hamiltonian framework the relation between conserved quantities and local one-parameter groups can be described as follows. The gradient $df$ of a function $f$ on phase space that Poisson-commutes with the Hamilton function $H$ is a one-form on phase space. The symplectic form $\Omega$ on the phase space is non-degenerate and allows us to identify vector fields and one-forms. In particular, we can associate to $df$ the unique vector field $\Gamma(df)$ for which $\Omega(\Gamma(df), X) = df(X)$ for any vector field $X$ on phase space.

Given the vector field $\Gamma(df)$ one can obtain a Hamiltonian phase flow, i.e. a local one-parameter group of diffeomorphisms $\varphi_t$ of the phase space which obey $\frac{d}{dt}\big|_{t=0}\varphi_t f = \Gamma(df)$. (A special case of this construction is the time evolution of the system, which is obtained by taking $f$ equal to the Hamilton function $H$.) The maps $\varphi_t$ are indeed symmetries of the system: $\varphi_t$ and the time evolution commute precisely if $f$ and $H$ Poisson-commute.

A more detailed investigation of symmetries in classical Hamiltonian systems or, more generally, group actions on symplectic manifolds, can be performed with the help of so-called moment maps. For a description of this concept we refer the reader to appendix 5 of [Arnold 1978] and to chapter 11 of [Marsden and Ratiu 1994].

### 1.6 Quantum mechanics

To analyze symmetries in quantum mechanics, we will work with canonical quantization in the Heisenberg picture. The basic idea of canonical quantization is to associate to any classical dynamical system in the Hamiltonian formulation a quantum dynamical system, which is accomplished by a certain quantization prescription. In short, this prescription amounts to declaring, first, that self-adjoint linear operators on some Hilbert space $\mathcal{H}$ over the complex numbers take over the rôle of the observables of the system from the functions on phase space; and second, to replacing the Poisson bracket of functions by the commutator of the associated operators on $\mathcal{H}$, according to

$$\{\cdot, \cdot\} \rightarrow \frac{i}{\hbar} [\cdot, \cdot]. \quad (1.21)$$

This leads in particular to the Heisenberg commutation relation between position and momentum operators:

$$\{p, q\} = 1 \rightarrow \frac{i}{\hbar} [p, q] = 1. \quad (1.22)$$

The number $\hbar$ appearing in the prescription (1.21) is Planck’s constant, which has the dimension of an action, and is present for dimensional reasons. The factor of $i \equiv \sqrt{-1}$ is needed in (1.21) because the Poisson bracket of two real functions is real again, while the commutator of two self-adjoint operators is anti-self-adjoint.
1.6 Quantum mechanics

It should be noted that the formula

\[ [A, B] := A \circ B - B \circ A \]  

(1.23)

defining the commutator of two operators \( A \) and \( B \) on \( \mathcal{H} \) is not as innocent as it may seem. The reason is that in quantum mechanics, observables are typically described by operators which are not defined on the whole Hilbert space \( \mathcal{H} \), but only on a dense subspace. As a consequence, the composition of operators, and hence also their commutator, is not necessarily defined. However, assuming that the commutator of the operators in question does exist, it is easy to check that it fulfills the same three properties that characterize the Poisson bracket of classical mechanics: it is bilinear,

\begin{align}
\text{a)} & \quad [\lambda A + \mu B, C] = \lambda [A, C] + \mu [B, C] ; \\
\text{b)} & \quad [A, B] = -[B, A] ; \\
\text{c)} & \quad [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0
\end{align}

(1.24)

it is antisymmetric,

(1.25)

and the Jacobi identity

(1.26)

holds. Again we can summarize these properties by the statement that the commutator \( [\cdot, \cdot] \) endows the space of linear operators on \( \mathcal{H} \) with the structure of a Lie algebra.

In the Heisenberg picture of quantum mechanics the physical states, i.e. the elements of the Hilbert space \( \mathcal{H} \), do not depend on time, but the observables do. Their time dependence is described by the Heisenberg equation

\[ \frac{d}{dt} A(t) = \frac{i}{\hbar} [H, A] \]

(1.27)

which is obtained by applying (1.22) to the classical formula (1.19). As a consequence, in quantum mechanics observables \( A \) that describe conservation laws are characterized by the fact that they commute with the Hamiltonian \( H \),

\[ [H, A] = 0 . \]

(1.28)

Again it follows from the Jacobi identity that the operators which satisfy (1.28) form a subalgebra. These statements are valid in the Schrödinger picture as well, as can be seen by employing the relation between operators \( A \equiv A_{(H)} \) in the Heisenberg picture and \( A_{(S)} \) in the Schrödinger picture, which reads

\[ A_{(H)}(t) = e^{iHt/\hbar} A_{(S)} e^{-iHt/\hbar} . \]

(1.29)
1 Symmetries and conservation laws

In quantum mechanics, too, the existence of conserved quantities constrains the dynamics; it implies selection rules for the matrix elements of the Hamiltonian. As an example, consider a system in which orbital angular momentum is conserved, as happens e.g. for the electron in the hydrogen atom when the spin degree of freedom is neglected. The eigenstates of $\mathbf{H}$ can then be labelled by their angular momentum $l$ and the projection $m$ of angular momentum on some coordinate axis, say the $z$-axis, together with some other quantum numbers, in the case of the hydrogen atom the principal quantum number $n$. The fact that $\mathbf{H}$ and $L_z$ commute then implies that

$$0 = \langle n', l', m' | [L_z, H] | n, l, m \rangle = (m' - m) \langle n', l', m' | H | n, l, m \rangle$$

(1.30)

for all eigenstates $| n, l, m \rangle$, $| n', l', m' \rangle$. As a consequence, the matrix element $\langle n', l', m' | H | n, l, m \rangle$ can be non-zero only if $m' = m$. This simple observation proves to be very powerful, and is the basis for many selection rules in physics (for some details, see chapter 16).

Suppose now that we are given a self-adjoint operator $A$ which commutes with the Hamiltonian $\mathbf{H}$. By exponentiation, one obtains operators

$$U_A(t) := e^{itA} \equiv 1 + itA + \frac{1}{2!}(itA)^2 + \ldots$$

(1.31)

for any $t \in \mathbb{R}$. These operators are unitary and form a group (see exercise 1.5), with multiplication law

$$U_A(t_1) \circ U_A(t_2) = U_A(t_1 + t_2) \quad \text{for all} \ t_1, t_2 \in \mathbb{R}. \quad (1.32)$$

Together all these unitary one-parameter groups generate a group of symmetries which is unitarily represented on the Hilbert space. The construction of this group is commonly referred to as ‘integrating up’ the ‘infinitesimal symmetries’ described by the operators $A$. As an example consider the momentum operator $\mathbf{p}$ which generates infinitesimal translations; in a position space description it is described by the differential operator $\mathbf{p} := -i \frac{\partial}{\partial x}$. One then finds (see exercise 1.4) that finite translations by a shift $a$ are described by the unitary operator

$$e^{iap/\hbar} = \exp(a \frac{\mathbf{p}}{\hbar}). \quad (1.33)$$

It must be noted that an exponentiation of infinitesimal symmetries to unitary operators as in (1.31) is not always possible, and that in the case of unbounded self-adjoint operators the exponential in (1.31) has to be defined using functional calculus (e.g. if the power series does not naively make sense). Particular care is required if there are infinitely many independent infinitesimal symmetries. The translations (1.33) are in this respect rather special even among finite-dimensional symmetries, because partial derivatives $\frac{\partial}{\partial x_j}$ commute. (Thus all commutators $[p_i, p_j]$