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978-0-521-54031-5 - Frobenius Algebras and 2D Topological Quantum Field Theories
Joachim Kock
Excerpt
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Introduction

In this introduction we briefly explain the words of the title of these notes, give a sketch of what we are going to do with these notions, and outline the viewpoint we will take in order to understand the structures. In the course of this introduction a lot of other words will be used which are probably no more familiar than those they are meant to explain – but don't worry: in the main text, all these words are properly defined and carefully explained . . .

Frobenius algebras. A Frobenius algebra is a finite-dimensional algebra equipped with a nondegenerate bilinear form compatible with multiplication. (Chapter 2 is all about Frobenius algebras.) Examples are matrix rings, group rings, the ring of characters of a representation, and Artinian Gorenstein rings (which in turn include cohomology rings, local rings of isolated hypersurface singularities . . .)

In algebra and representation theory such algebras have been studied for a century, along with various related notions – see Curtis and Reiner [15].

Frobenius structures. During the past decade, Frobenius algebras have shown up in a variety of topological contexts, in theoretical physics and in computer science. In physics, the main scenery for Frobenius algebras is that of topological quantum field theory (TQFT), which in its axiomatisation amounts to a precise mathematical theory. In computer science, Frobenius algebras arise in the study of flowcharts, proof nets, circuit diagrams . . .

In any case, the reason Frobenius algebras show up is that this is essentially a topological structure: it turns out that the axioms for a Frobenius algebra can be given completely in terms of graphs – or as we shall do, in terms of topological surfaces.

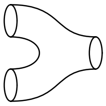
Frobenius algebras are just algebraic representations of this structure – the goal of these notes is to make all this precise. We will focus on topological quantum field theories – and in particular on dimension 2. This is by far the best picture of Frobenius structures since the topology is explicit, and since there is no additional structure to complicate things. In fact, the main theorem of these notes states that there is an equivalence of categories between that of 2-dimensional TQFTs and that of commutative Frobenius algebras.

(There will be no further mention of computer science in these notes.)

Topological quantum field theories. In the axiomatic formulation (due to Atiyah [5]), an n-dimensional topological quantum field theory is a rule $\mathscr{A}$ which to each closed oriented manifold Σ (of dimension $n-1$) associates a vector space $\Sigma\mathscr{A}$, and to each oriented n-manifold whose boundary is Σ associates a vector in $\Sigma\mathscr{A}$. This rule is subject to a collection of axioms which express that topologically equivalent manifolds have isomorphic associated vector spaces, and that disjoint unions of manifolds go to tensor products of vector spaces, etc.

Cobordisms. The clearest formulation is in categorical terms. First one defines a category of cobordisms ***nCob***: the objects are closed oriented $(n-1)$-manifolds, and an arrow from Σ to Σ' is an oriented n-manifold M whose 'in-boundary' is Σ and whose 'out-boundary' is Σ'. (The cobordism M is defined up to diffeomorphism rel the boundary.) The simplest example of a cobordism is the cylinder $\Sigma \times I$ over a closed manifold Σ, say a circle. It is a cobordism from one copy of Σ to another.

Here is a drawing of a cobordism from the union of two circles to one circle:



Composition of cobordisms is defined by gluing together the underlying manifolds along common boundary components; the cylinder $\Sigma \times I$ is the identity arrow on Σ. The operation of taking disjoint union of manifolds and cobordisms gives this category *monoidal structure* – more about monoidal categories later. On the other hand, the category ***Vect***$_{\Bbbk}$ of vector spaces is monoidal under tensor products.

Now the axioms amount to saying that a TQFT is a (symmetric) monoidal functor from ***nCob*** to ***Vect***$_{\Bbbk}$. This is also called a linear representation of ***nCob***.

So what does this have to do with Frobenius algebras? Before we come to the relation between Frobenius algebras and 2-dimensional TQFTs, let us make a couple of remarks on the motivation for TQFTs.

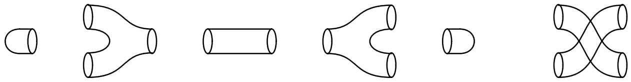
Physical interest in TQFTs comes mainly from the observation that TQFTs possess certain features one expects from a theory of quantum gravity. It serves as a baby model in which one can do calculations and gain experience before embarking on the quest for the full-fledged theory, which is expected to be much more complicated. Roughly, the closed manifolds represent *space*, while the cobordisms represent *space-time*. The associated vector spaces are then the *state spaces*, and an operator associated to a space-time is the time-evolution operator (also called transition amplitude, or Feynman path integral). That the theory is topological means that the transition amplitudes do not depend on any additional structure on space-time (like Riemannian metric or curvature), but only on the topology. In particular there is no time-evolution along cylindrical space-time. That disjoint union goes to tensor product expresses the common principle in quantum mechanics that the state space of two independent systems is the tensor product of the two state spaces.

(No further explanation of the relation to physics will be given – the author of these notes recognises he knows nearly nothing of this aspect. The reader is referred to Dijkgraaf [17] or Barrett [11], for example.)

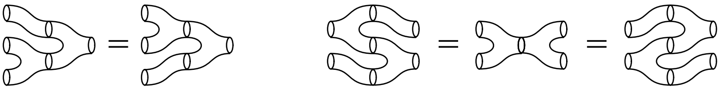
Mathematical interest in TQFTs stems from the observation that they produce invariants of closed manifolds: an n-manifold without boundary is a cobordism from the empty $(n-1)$-manifold to itself, and its image under $\mathscr{A}$ is therefore a linear map $\Bbbk \to \Bbbk$, i.e. a scalar. It was shown by Witten how TQFT in dimension 3 is related to invariants of knots and the Jones polynomial – see Atiyah [6].

The viewpoint of these notes is different however: instead of developing TQFTs in order to describe and classify manifolds, we work in dimension 2 where a complete classification of surfaces already exists; we then use this classification to describe TQFTs!

Cobordisms in dimension 2. In dimension 2, 'everything is known': since surfaces are completely classified, one can also describe the cobordism category completely. Every cobordism is obtained by composing the following basic building blocks (each with the in-boundary drawn to the left):



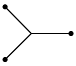
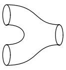
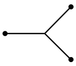
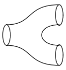
Two connected cobordisms are equivalent if they have the same genus and the same number of in- and out-boundaries. This gives a bunch of relations, and a complete description of the monoidal category ***2Cob*** in terms of generators and relations. Here are two examples of relations that hold in ***2Cob***:



(1)

These equations express that certain surfaces are topologically equivalent rel the boundary.

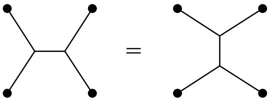
Topology of some basic algebraic operations. Some very basic principles are in play here: 'creation', 'coming together', 'splitting up', 'annihilation'. These principles have explicit mathematical manifestations as algebraic operations:

Principle	Feynman diagram	2D cobordism	Algebraic operation (in a $\Bbbk$-algebra A)	
merging			multiplication	$A \otimes A \to A$
creation			unit	$\Bbbk \to A$
splitting			comultiplication	$A \to A \otimes A$
annihilation			counit	$A \to \Bbbk$

Note that in the intuitive description there is a notion of time involved which accounts for the distinction between coming-together and splitting-up – or perhaps 'time' is too fancy a word, but at least there is a notion of *start* and *finish*. Correspondingly, in the algebraic or categorical description the notion of morphism involves a direction: morphisms are *arrows*, and they have well defined *source* and *target*.

It is an important observation from category theory that many algebraic structures admit descriptions purely in terms of arrows (instead of referring to elements) and commutative diagrams (instead of equations among elements). In particular, this is true for the notion of an *algebra*: an algebra is a vector space A equipped with two maps $A \otimes A \to A$ and $\Bbbk \to A$, satisfying the associativity axiom and the unit axiom. Now according to the above dictionary, the left-hand relation of Equation (1) is just the topological expression of associativity! Put in other words, the associativity equation has topological content: it expresses the topological equivalence of two surfaces (or two graphs).

It gives sense to other operations, like merging (or splitting) three particles: it makes no difference whether we first merge two of them and then merge the result with the third, or whether we merge the last two with the first. From the viewpoint of graphs, the basic axiom (equivalent to Equation (1)) is that two vertices can move past each other:



Frobenius algebras. In order to relate this to Frobenius algebras the definition given in the beginning of this introduction is not the most convenient. It turns out one can characterise a Frobenius algebra as follows: it is an algebra (multiplication denoted ) which is simultaneously a *coalgebra* (comultiplication denoted ) with a certain compatibility condition between  and . This compatibility condition is exactly the right-hand relation drawn in Equation (1). (Note that by the dictionary, this is just a graphical expression of a precise algebraic requirement.) In fact, the relations that hold in ***2Cob*** correspond precisely to the axioms of a commutative Frobenius algebra. This comparison leads to the main theorem:

Theorem. *There is an equivalence of categories*

$$\textbf{\textit{2TQFT}} \simeq \textbf{\textit{cFA}},$$

given by sending a TQFT to its value on the circle (the unique closed connected 1*-manifold).*

So in this sense, we can say, if we want, that Frobenius algebras are the same thing as linear representations of ***2Cob***.

The idea of the proof is this: let A be the image of the circle, under a TQFT $\mathscr{A}$. Now $\mathscr{A}$ sends each of the generators of ***2Cob*** to a linear map between tensor powers of A, just as tabulated above. The relations which hold in ***2Cob*** are preserved by $\mathscr{A}$ (since $\mathscr{A}$ by definition is a monoidal functor) and in its target category ***Vect*** they translate into the axioms for a commutative Frobenius algebra! (Conversely, every commutative Frobenius algebra can be used to define a 2-dimensional TQFT.)

Monoidal categories. As mentioned, just in order to define the category ***TQFT*** we need the notion of monoidal categories. In fact, monoidal categories is the best framework to understand all the concepts described above. The notion of associative multiplication with unit is precisely what the abstract concept of *monoid* encodes – and monoids live in monoidal categories.

The prime example of a monoidal category is the category $\textbf{\textit{Vect}}_{\Bbbk}$ of vector spaces and tensor products, with the ground field as neutral object. In general a monoidal category is a category equipped with some sort of 'product' like $\otimes$ or $\coprod$, satisfying certain properties. This 'product' serves as background for defining the multiplication maps, i.e. defining monoids: a monoid in $(\textbf{\textit{Vect}}_{\Bbbk}, \otimes, \Bbbk)$ is precisely a $\Bbbk$-algebra A, since the multiplication map is described as a $\Bbbk$-linear map $A \otimes A \to A$, etc. Another example of a monoid is the circle in ***2Cob***. . .

The simplex category Δ and what it means to monoids and algebras. There is a little monoidal category which bears some similarity with ***2Cob***: the simplex category Δ is roughly the category of finite ordered sets and order-preserving maps. It is a monoidal category under disjoint union. To be more precise, the objects of Δ are $\mathbf{n} = \{0, 1, 2, \ldots, n-1\}$, one for each $n \in \mathbb{N}$, and the arrows are the maps $f : \mathbf{m} \to \mathbf{n}$ such that $i \leq j \Rightarrow if \leq jf$. There are several other descriptions of this important category – one is in graphical terms, and reveals it as a subcategory of ***2Cob***. The object $\mathbf{1}$ is a monoid in Δ, and in a sense Δ is the smallest possible monoidal category which contains a nontrivial monoid. In fact the following universal property is shown to hold: *every monoid in any monoidal category* $\mathbf{V}$ *is the image of* $\mathbf{1}$ *under a unique monoidal functor* $\Delta \to \mathbf{V}$. This is to say that Δ is the free monoidal category containing a monoid. In particular, $\Bbbk$-algebras can be interpreted as 'linear representations' of Δ.

Observing that Δ can be described graphically, we see that this result is of exactly the same type as our main Theorem.

Frobenius objects. Once we have taken the step of abstraction from $\Bbbk$-algebras to monoids in an arbitrary monoidal category, it is straightforward to define the notion of Frobenius object in a monoidal category: it is an object equipped with four maps as those listed in the table, and with the compatibility condition expressed in Equation (1). In certain monoidal categories, called symmetric, it makes sense to ask whether a monoid or a Frobenius object is commutative, and of course these notions are defined in such a way that commutative Frobenius objects in $\textbf{\textit{Vect}}_{\Bbbk}$ are precisely commutative Frobenius algebras.

Universal Frobenius structure. With these general notions, generalisation of the Theorem is immediate: all the arguments of the proof do in fact carry over to the setting of an arbitrary (symmetric) monoidal category, and we find that ***2Cob*** is the free symmetric monoidal category containing a commutative Frobenius object. This means that *every commutative Frobenius object in any symmetric monoidal category* $\mathbf{V}$ *is the image of the circle under a unique symmetric monoidal functor from* ***2Cob***.

Since the proof of this result is the same as the proof of the original theorem, this is the natural generality of the statement. The interest in this generality is that it actually includes many natural examples of TQFTs which could not fit into the original definition. For example, in our treatment of Frobenius algebras in Chapter 2 we will see that cohomology rings are Frobenius algebras in a natural way, but typically they are not commutative but only graded-commutative. For this reason they cannot support a TQFT in the strict sense. But if instead of the usual symmetric monoidal category ***Vect*** we take for example the category of graded vector spaces with 'super-symmetry' structure, then all cohomology rings can support a TQFT (of this slightly generalised sort).

It is the good generalised version of the main theorem that makes this clear. In many sources on TQFTs, the questions of symmetry are swept under the carpet and the point about 'super-symmetric' TQFTs is missed.

In these notes, the whole question of symmetry is given a rather privileged rôle. The difficult thing about symmetry is to avoid mistaking it for identity! For example, for the cartesian product $\times$ (which is an important example of a monoidal structure), it is *not* true that $X \times Y = Y \times X$. What is true is that there is a natural isomorphism between the two sets (or spaces). Similar observations are due for disjoint union $\coprod$, and tensor product $\otimes$. . . While it requires some pedantry to treat symmetry properly, it is necessary in order to understand the super-symmetric examples just mentioned.

Organisation of these notes. The notes are divided into three chapters each of which should be read before the others! The first chapter is about topology – cobordisms and TQFTs; Chapter 2 is about algebra – Frobenius algebras; and Chapter 3 is mostly category theory. The reader is referred to the Contents for more details on where to find what.

Although the logical order of the material is not completely linear, hopefully the order is justified pedagogically: we start with geometry! – the concrete and palpable – and then we gradually proceed to more abstract subjects (or should we say: more abstract aspects of our subject), helped by drawings and intuition provided by the geometry. With the experience gained with these investigations we get ready to try to understand the abstract structures behind. The ending is about very abstract concepts and objects with universal properties, but we can cope with that because we know the underlying geometry – in fact we show that this very abstract thing with that universal property is precisely the cobordism category we described so carefully in Chapter 1.

Exercises. Each section ends with a collection of exercises of varying level and interest. Most of them are really easy, and the reader is encouraged to do them all. A few of them are considered less straightforward and have been marked with a star.

Further reading. My great sorrow about these notes is that I do not understand the physical background or interpretation of TQFTs. The physically inclined reader must resort to the existing literature, for example Atiyah's book [6] or the notes of Dijkgraaf [17]. I would also like to recommend John Baez's web site [8], where a lot of references can be found.

Within the categorical viewpoint, an important approach to Frobenius structures which has not been touched upon is the 2-categorical viewpoint, in terms of monads and adjunctions. This has recently been exploited to great depth by Müger [39]. Again, a pleasant introductory account is given by Baez [8], TWF 174 (and 173).

Last but not least, I warmly recommend the lecture notes of Quinn [43], which are detailed and go in depth with concrete topological quantum field theories.

1

Cobordisms and topological quantum field theories

Summary

In the first section we recall some basic notions of manifolds with boundary and orientations, and Morse functions. We introduce the slightly nonstandard notion of *in-boundary* and *out-boundary*, which is particularly convenient for the treatment of cobordisms.

Section 1.2 is devoted to the basic theory of oriented cobordisms. Roughly a cobordism between two closed $(n-1)$-manifolds is an n-manifold whose boundary is made up of the two $(n-1)$-manifolds. We describe what it means for two cobordisms to be equivalent. Next we introduce the *decomposition* of a cobordism, which amounts to cutting up along a closed codimension-1 submanifold, obtaining two cobordisms. Finally we state the axioms for a topological quantum field theory (TQFT) in the style of Atiyah [5]: it is a way of associating vector spaces and linear maps to $(n-1)$-manifolds and cobordisms, respecting decompositions and disjoint union. A special decomposition of the cylinder shows that a vector space which is image of a TQFT comes equipped with a nondegenerate bilinear pairing, in a strong sense, which in particular forces the vector space to be finite dimensional.

In Section 1.3 we assemble the manifolds and cobordisms into a category ***nCob***. In order to have a well defined composition we must pass to a quotient, identifying equivalent cobordisms. The identity arrows are the cylinder classes. Then we start discussing the monoidal structure: disjoint union of cobordisms. With this terminology we can define a TQFT as a (symmetric) monoidal functor from ***nCob*** to $\textbf{\textit{Vect}}_{\Bbbk}$. (The definition and basic properties of monoidal categories are given in Chapter 3.)

Finally in Section 1.4 we specialise to dimension 2. Here we can give a complete description of ***2Cob*** in terms of generators and relations for a monoidal category. These results depend on the classification theorem for topological surfaces.

The notion of cobordism goes back to Pontryagin and Thom [47] in the 1950s. Topological quantum field theories were introduced by Witten [50], and the mathematical axiomatisation was soon after proposed by Atiyah [5] (1989). The description of ***2Cob*** in terms of generators and relations was first given explicitly by Abrams [1] (1995), but the proof is already sketched in Quinn [43], and most likely it goes further back.

1.1 Geometric preliminaries

The reader is expected to be familiar with the most basic notions of differentiable manifolds, their tangent bundles, smooth maps, and their differentials. Our main reference will be Hirsch [27]. A more elementary introduction which emphasises the concepts used here is Wallace [48]. In this section we just collect some crucial notions, establish terminology and notation, and give some basic examples which we will need later on.

Our manifolds are not assumed to be embedded in Euclidean space; an n-manifold is merely a topological space M covered by open sets homeomorphic to $\mathbb{R}^n$. These maps are called coordinate charts, and the collection of all of them is called an atlas. *Smooth* means differentiable of class C^∞; that is, on overlaps between two such charts, the coordinate change functions are differentiable maps of class C^∞ (between subsets of $\mathbb{R}^n$). A smooth structure on M is a maximal smooth atlas. Throughout, *manifold* will mean smooth manifold, i.e. a manifold equipped with a smooth structure. All our manifolds will be compact, but we do not assume them to be connected.

Let us note that we regard the empty set as an n-manifold! – we denote it $\emptyset_n$. This is justified by the observation that every point of $\emptyset_n$ has a neighbourhood homeomorphic to $\mathbb{R}^n$.

Manifolds with boundary

In a usual n-manifold M, every point x has a neighbourhood homeomorphic to $\mathbb{R}^n$. This means that from x you can move a little bit in any direction. This is possible either because M 'curves back and closes up itself' like a circle, a sphere, or a torus, or because M is open, like for example the open disc $\{x \in \mathbb{R}^2 \mid |x| < 1\}$: here the points are not allowed to sit on the boundary, so no matter how close you are to it you can always come a little bit closer.

In a manifold with boundary the 'boundary points' are included, like for example the points on the circumference in $\{x \in \mathbb{R}^2 \mid |x| \leq 1\}$. For such a point, there are directions in which it is impossible to move: there is no neighbourhood homeomorphic to $\mathbb{R}^n$, so we need a new sort of chart. It is practical