Introduction

Fourier and Laplace transforms are examples of mathematical operations which can play an important role in the analysis of mathematical models for problems originating from a broad spectrum of fields. These transforms are certainly not new, but the strong development of digital computers has given a new impulse to both the applications and the theory. The first applications actually appeared in astronomy, prior to the publication in 1822 of the famous book *Théorie analytique de la chaleur* by Joseph Fourier (1768 – 1830). In astronomy, sums of sine and cosine functions were already used as a tool to describe *periodic* phenomena. However, in Fourier's time one came to the surprising conclusion that the Fourier theory could also be applied to *non-periodic* phenomena, such as the important physical problem of heat conduction. Fundamental for this was the discovery that an arbitrary function could be represented as a superposition of sine and cosine functions, hence, of simple periodic functions. This also reflects the essential starting point of the various Fourier and Laplace transforms: to represent functions or signals as a sum or an integral of simple functions or signals. The information thus obtained turns out to be of great importance for several applications. In electrical networks, for example, the sinusoidal voltages or currents are important, since these can be used to describe the operation of such a network in a convenient way. If one now knows how to express the voltage of a voltage source in terms of these sinusoidal signals, then this information often enables one to calculate the resulting currents and voltages in the network.

Applications of Fourier and Laplace transforms occur, for example, in physical problems, such as heat conduction, and when analyzing the transfer of signals in various systems. Some examples are electrical networks, communication systems, and analogue and digital filters. Mechanical networks consisting of springs, masses and dampers, for the production of shock absorbers for example, processes to analyze chemical components, optical systems, and computer programs to process digitized sounds or images, can all be considered as systems for which one can use Fourier and Laplace transforms as well. The specific Fourier and Laplace transform being used may differ from application to application. For electrical networks the Fourier and Laplace transforms are applied to functions describing a current or voltage as function of *time*. In heat conduction problems, transforms occur that are applied to, for example, a temperature distribution as a function of *position*. In the modern theory of digital signal processing, *discrete* versions of the Fourier and Laplace transforms are used to analyze and process a sequence of measurements or data, originating for example from an audio signal or a digitized photo.

In this book the various transforms are all treated in detail. They are introduced in a mathematically sound way, and many mutually related properties are derived, so that the reader may experience not only the differences, but above all the great coherence between the various transforms.

As a link between the various applications of the Fourier and Laplace transforms, we use the theory of signals and systems as well as the theory of ordinary and partial

FIGURE 0.1
When digitizing a photo, information is lost. Conditions under which a good reconstruction can be obtained will be discussed in part 5. *Copyright*: Archives de l'Académie des Sciences de Paris, Paris

differential equations. We do not assume that the reader is familiar with systems theory. It is, however, an advantage to have some prior knowledge of some of the elementary properties of linear differential equations.

Considering the importance of the applications, our first chapter deals with *signals* and *systems*. It is also meant to incite interest in the theory of Fourier and Laplace transforms. Besides this, part 1 also contains a chapter with mathematical preparations for the parts to follow. Readers with a limited mathematical background are offered an opportunity here to supplement their knowledge.

In part 2 we meet our first transform, specifically meant for *periodic* functions or signals. This is the theory of *Fourier series*. The central issue in this part is to investigate the information on a periodic function that is contained in the so-called Fourier coefficients, and especially if and how a periodic function can be described by these Fourier coefficients. The final chapter of this part examines some of the applications of Fourier series in continuous-time systems and in solving ordinary and partial differential equations. Differential equations often originate from a physical problem, such as heat conduction, or from electrical networks.

Part 3 treats the *Fourier integral* as a transform that is applied to functions which are *no longer periodic*. In order to construct a sound theory for the Fourier integral – keeping the applications in mind – we can no longer content ourselves with the classical notion of a function. In this part we therefore pay special attention in chapters 8 and 9 to *distributions*, among which is the well-known delta function. Usually, a consistent treatment of the theory of distributions is only found in advanced textbooks on mathematics. This book shows that a satisfactory treatment is also feasible for readers without a background in theoretical mathematics. In the final chapter of this part, the use of the Fourier integral in systems theory and in solving partial differential equations is explained in detail.

The *Laplace transform* is the subject of part 4. This transform is particularly relevant when we are dealing with phenomena that are *switched on*. In the first chapter an introduction is given to the theory of complex functions. It is then easier for the reader to conceive of a Laplace transform as a function defined on the complex numbers. The treatment in part 4 proceeds more or less along the same lines as in parts 2 and 3, with a focus on the applications in systems theory and in solving differential equations in the closing chapter.

In parts 2, 3 and 4, transforms were considered for functions defined on the real numbers or on a part of these real numbers. Part 5 is dedicated to the *discrete* transforms, which are intended for functions or signals defined on the integers.

These functions or signals may arise by *sampling* a continuous-time signal, as in the digitization of an audiosignal (or a photo, as in figure 0.1). In the first chapter of this part, we discuss how this can be achieved without loss of information. This results in the important sampling theorem. The second chapter in this part starts with the treatment of the first discrete transform in this book, which is the so-called *discrete Fourier transform*, abbreviated as DFT. The *Fast Fourier Transform*, abbreviated as FFT, is the general term for several fast algorithms to calculate the DFT numerically. In the third chapter of part 5 an FFT, based on the popular situation where the 'length of the DFT' is a power of two, is treated extensively. In part 5 we also consider the z-transform, which plays an important role in the analysis of discrete systems. The final chapter is again dedicated to the applications. This time, the use of discrete transforms in the study of discrete systems is explained.

Part 1

Applications and foundations

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Excerpt
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CHAPTER 1

Signals and systems

INTRODUCTION

Fourier and Laplace transforms provide a technique to solve differential equations which frequently occur when translating a physical problem into a mathematical model. Examples are the vibrating string and the problem of heat conduction. These will be discussed in chapters 5, 10 and 14.

Besides solving differential equations, Fourier and Laplace transforms are important tools in analyzing signals and the transfer of signals by systems. Hence, the Fourier and Laplace transforms play a predominant role in the theory of signals and systems. In the present chapter we will introduce those parts of the theory of signals and systems that are crucial to the application of the Fourier and Laplace transforms. In chapters 5, 10, 14 and 19 we will then show how the Fourier and Laplace transforms are utilized.

Signals and systems are introduced in section 1.1 and then classified in sections 1.2 and 1.3, which means that on the basis of a number of properties they will be divided into certain classes that are relevant to applications. The fundamental signals are the sinusoidal signals (i.e. sine-shaped signals) and the time-harmonic signals. Time-harmonic signals are complex-valued functions (the values of these functions are complex numbers) which contain only one frequency. These constitute the fundamental building blocks of the Fourier and Laplace transforms.

The most important properties of systems, treated in section 1.3, are linearity and time-invariance. It is these two properties that turn Fourier and Laplace transforms into an attractive tool. When a linear time-invariant system receives a time-harmonic signal as input, the resulting signal is again a time-harmonic signal with the same frequency. The way in which a linear time-invariant system transforms a time-harmonic signal is expressed by the so-called frequency response, which will also be considered in section 1.3.

The presentation of the theory of signals and systems, and of the Fourier and Laplace transforms as well, turns out to be much more convenient and much simpler if we allow the signals to have complex numbers as values, even though in practice the values of signals will usually be real numbers. This chapter will therefore assume that the reader has some familiarity with the complex numbers; if necessary one can first consult part of chapter 2, where the complex numbers are treated in more detail.

LEARNING OBJECTIVES
After studying this chapter it is expected that you
- know what is meant by a signal and a system
- can distinguish between continuous-time, discrete-time, real, complex, periodic, power, energy and causal signals
- know what a sinusoidal and a time-harmonic signal are
- are familiar with the terms amplitude, frequency and initial phase of a sinusoidal and a time-harmonic signal
- know what is meant by the power- and energy-content of a signal and in particular know what the power of a periodic signal is
- can distinguish between continuous-time, discrete-time, time-invariant, linear, real, stable and causal systems
- know what is meant by the frequency response, amplitude response and phase response for a linear time-invariant system
- know the significance of a sinusoidal signal for a real linear time-invariant system
- know the significance of causal signals for linear time-invariant causal systems.

1.1 Signals and systems

To clarify what will be meant by signals and systems in this book, we will first consider an example.

In figure 1.1 a simple electric network is shown in which we have a series connection of a resistor R, a coil L and a voltage generator. The generator in the network

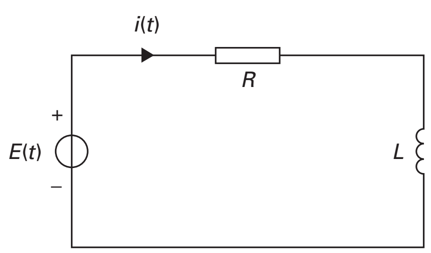

FIGURE 1.1
Electric network with resistor, coil and voltage generator.

supplies a voltage $E(t)$ and as a consequence a current $i(t)$ will flow in the network. From the theory of electrical networks it follows that the current $i(t)$ is determined unambiguously by the voltage $E(t)$, assuming that before we switch on the voltage generator, the network is at rest and hence there is no current flowing through the coil and resistor. We say that the current $i(t)$ is uniquely determined by the voltage $E(t)$. Using the Kirchhoff voltage-law and the current–voltage relationship for the resistor R and coil L, one can derive an equation from which the current $i(t)$ can be calculated explicitly as a function of time. Here we shall not be concerned with this derivation and merely state the result:

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} e^{-(t-\tau)R/L} E(\tau)\, d\tau. \tag{1.1}$$

This is an integral relationship of a type that we shall encounter quite frequently in this book. The causal relation between $E(t)$ and $i(t)$ can be represented by the diagram of figure 1.2. The way in which $i(t)$ follows from $E(t)$ is thus given by the

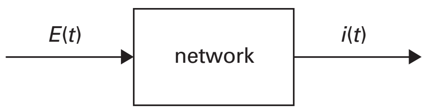

FIGURE 1.2
The relation between $E(t)$ and $i(t)$.

relation (1.1). Mathematically, this can be viewed as a mapping which assigns to a function $E(t)$ the function $i(t)$. In systems theory this mapping is called a system. The functions $E(t)$ and $i(t)$ are called the input and output respectively.

So a system is determined once the relationship is known between input and corresponding output. It is of no importance how this relationship can be realized physically (in our example by the electrical network). Often a system can even be realized in several ways. To this end we consider the mechanical system in figure 1.3, where a point-mass P with mass m is connected by an arm to a damper D. The point-mass P is acted upon by a force $F(t)$. As a result of the force the point P

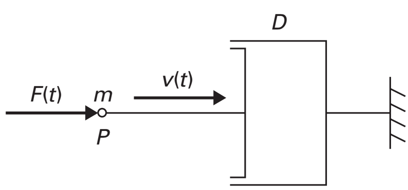

FIGURE 1.3
Mechanical system.

moves with velocity $v(t)$. The movement causes a frictional force K in the damper which is proportional to the velocity $v(t)$, but in direction opposite to the direction of $v(t)$. Let k be the proportionality constant (the damping constant of the damper), then $K = -kv(t)$. Using Newton's law one can derive an equation of motion for the velocity $v(t)$. Given $F(t)$ one can then obtain from this equation a unique solution for the velocity $v(t)$, assuming that when the force $F(t)$ starts acting, the mechanical system is at rest. Again we shall not be concerned with the derivation and only state the result:

$$v(t) = \frac{1}{m}\int_{-\infty}^{t} e^{-(t-\tau)k/m} F(\tau)\,d\tau. \tag{1.2}$$

Relation (1.2) defines, in the same way as relation (1.1), a system which assigns to an input $F(t)$ the output $v(t)$. But when $R = k$ and $L = m$ then, apart from the dimensions of the physical quantities involved, relations (1.1) and (1.2) are identical and hence the systems are equal as well. The realizations however, are different!

This way of looking at systems has the advantage that the properties which can be deduced from a system apply to all realizations. This will in particular be the case for the applications of the Fourier and Laplace transforms.

It is now the right moment to introduce the concept of a signal. The previous examples give rise to the following description of the notion of a signal.

Signal

A *signal* is a function.

Thus, in the example of the electrical network, the voltage $E(t)$ is a signal, which is defined as a function of time. The preceding description of the concept of a signal is very general and has thus a broad application. It merely states that it is a function. Even the domain, the set on which the function is defined, and the range, the set of function-values, are not prescribed. For instance, the yearly energy consumption in the Netherlands can be considered as a signal. See figure 1.4.

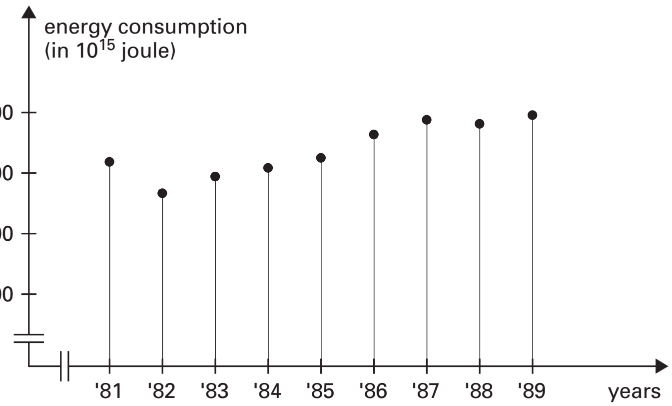

FIGURE 1.4
Energy consumption in the Netherlands.

Now that we have introduced the notion of a signal, it will also be clear from the foregoing what the concept of a system will mean in this book.

System

A *system* is a mapping L assigning to an *input* u a unique *output* y.

Response

It is customary to represent a system as a 'black box' with an input and an output (see figure 1.5). The output y corresponding to the input u is uniquely determined by u and is called the *response* of the system to the input u.

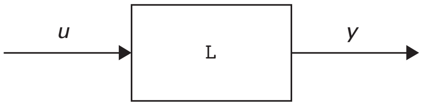

FIGURE 1.5
System.

When y is the response of a system L to the input u, then, depending on the context, we use either of the two notations

$$y = \mathtt{L}u,$$
$$u \mapsto y.$$

Our description of the concept of a system allows only one input and one output. In general more inputs and outputs are possible. In this book we only consider systems with one input and one output.

In the next section, signals will be classified on the basis of a number of properties.

1.2 Classification of signals

Real signal

The values that a signal can attain will in general be real numbers. This has been the case in all previous examples. Such signals are called *real* or *real-valued signals*. However, in the treatment of Fourier and Laplace transforms it is a great advantage to work with signals that have complex numbers as values. This means that we will suppose that a signal f has the form

$$f = f_1 + if_2,$$

Complex signal

where i is the imaginary unit for which $i^2 = -1$, and f_1 and f_2 are two real-valued signals. The signal f_1 is called the real part of the *complex signal* f (notation $\operatorname{Re} f$) and f_2 the imaginary part (notation $\operatorname{Im} f$). If necessary, one can first consult chapter 2, where a review of the theory of complex numbers can be found. In section 1.2.2 we will encounter an important example of a complex signal, the so-called time-harmonic signal.

Note that two complex signals are equal if the real parts and the imaginary parts of the complex signals agree. When for a signal f one has that $f_2 = \operatorname{Im} f = 0$, then the signal is real. When $f_1 = \operatorname{Re} f = 0$ and $f_2 = \operatorname{Im} f = 0$, then the signal f is equal to zero. This signal is called the *null-signal*.

Null-signal

Usually, the signals occurring in practice are real. Hence, when dealing with results obtained from the application of Fourier and Laplace transforms, it will be important to consider specifically the consequences for real signals.

1.2.1 Continuous-time and discrete-time signals

Continuous-time signal

In electrical networks and mechanical systems, the signals are a function of the time-variable t, a real variable which may assume all real values. Such signals are called *continuous-time* signals. However, it is not necessary that the adjective continuous-time has any relation with time as a variable. It only expresses the fact that the function is defined on $\mathbb{R}$ or a subinterval of $\mathbb{R}$. Hence, a continuous-time signal is a function defined on $\mathbb{R}$ or a subinterval of $\mathbb{R}$. One should not confuse the concept of a continuous-time signal with the concept of a continuous function as it is used in mathematics.

Discrete-time signal

In the example of the yearly energy consumption in the Netherlands, the signal is not defined on $\mathbb{R}$, but only defined for discrete moments of time. Such a signal can be considered as a function defined on a part of $\mathbb{Z}$, which is the set of integers. In our example the value at $n \in \mathbb{Z}$ is the energy consumption in year n. A signal defined on $\mathbb{Z}$, or on a part of $\mathbb{Z}$, will be called a *discrete-time signal*.

As a matter of fact we assume in this book, unless explicitly stated otherwise, that continuous-time signals are defined on the whole of $\mathbb{R}$ and discrete-time signals on the whole of $\mathbb{Z}$. In theory, a signal can always be extended to, respectively, the whole of $\mathbb{R}$ or the whole of $\mathbb{Z}$.

We denote continuous-time signals by $f(t)$, $g(t)$, etc. and discrete-time signals by $f[n]$, $g[n]$, etc., hence using square brackets surrounding the argument n.

The introduction of continuous-time and discrete-time signals that we have given above excludes functions of more than one variable. In this book we thus confine ourselves to signals depending on one variable only. As a consequence we also confine ourselves to systems where the occurring signals depend on one variable only.