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PART I

QUANTUM OPTICS AND QUANTUM INFORMATION

1 The quantum theory of light

Classically, light is an electromagnetic phenomenon, described by Maxwell's equations. However, under certain conditions, such as low intensity or in the presence of certain nonlinear optical materials, light starts to behave differently, and we have to construct a 'quantum theory of light'. We can exploit this quantum behaviour of light for quantum information processing, which is the subject of this book. In this chapter, we develop the quantum theory of the free electromagnetic quantum field. This means that we do not yet consider the interaction between light and matter; we postpone that to Chapter 7. We start from first principles, using the canonical quantization procedure in the Coulomb gauge: we derive the field equations of motion from the classical Lagrangian density for the vector potential, and promote the field and its canonical momentum to operators and impose the canonical commutation relations. This will lead to the well-known creation and annihilation operators, and ultimately to the concept of the photon. After quantization of the free electromagnetic field we consider transformations of the mode functions of the field. We will demonstrate the intimate relation between these linear mode transformations and the effect of beam splitters, phase shifters, and polarization rotations, and show how they naturally give rise to the concept of squeezing. Finally, we introduce coherent and squeezed states.

The first two sections of this chapter are quite formal, and a number of subtleties arise when we quantize the electromagnetic field, such as the continuum of modes, the gauge freedom, and the definition of the creation and annihilation operators with respect to the classical modes. Readers who have not encountered field quantization procedures before may find these sections somewhat daunting, but most of the subtleties encountered here have very little bearing on the later chapters. We mainly include the full derivation from first principles to give the field of optical quantum information processing a proper physical foundation, and derive the annihilation and creation operators of the discrete optical modes from the continuum of modes that is the electromagnetic field.

1.1 The classical electromagnetic field

Classical electrodynamics is the theory of the behaviour of electric and magnetic fields in the presence of charge and current distributions. It was shown by James Clerk Maxwell (1831–1879) that the equations of motion for electric and magnetic fields, the Maxwell equations, allow for electromagnetic waves. In vacuum, these waves propagate with a velocity $c = 299\,792\,458\ \mathrm{ms}^{-1}$, and Maxwell therefore identified these waves in a certain frequency range with light. In this section, we define the electric and magnetic fields in terms of the scalar and vector potentials, and construct the field Lagrangian density in the presence of charge and current distributions. Variation of the Lagrangian density with respect to the potentials then leads to Maxwell's equations. Subsequently, we consider the Maxwell equations for the vacuum, and derive the wave equation and its plane-wave solutions. The source-free Lagrangian density is then used to define the canonical momenta to the potentials, which in turn allow us to give the Hamiltonian density for the free field. These are the ingredients we need for the canonical quantization procedure in Section 1.2.

The electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are related to a scalar and a vector potential $\Phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \text{and} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)\,. \tag{1.1}$$

The most elegant way to construct a classical field theory is via the Lagrangian density. We can use the potentials as the dynamical variables of our classical field theory, which means that we can write the Lagrangian density $\mathscr{L}$ as a function of the potentials and their time derivatives

$$\mathscr{L} = \mathscr{L}(\Phi, \dot{\Phi}; \mathbf{A}, \dot{\mathbf{A}})\,. \tag{1.2}$$

The equations of motion for the potentials Φ and $\mathbf{A}$ are then given by the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\delta\mathscr{L}}{\delta\dot{\Phi}} - \frac{\delta\mathscr{L}}{\delta\Phi} = 0 \tag{1.3}$$

and

$$\frac{d}{dt}\frac{\delta\mathscr{L}}{\delta\dot{A}_j} - \frac{\delta\mathscr{L}}{\delta A_j} = 0\,. \tag{1.4}$$

Here δ denotes the functional derivative, since the potentials are themselves functions of space and time, and each component of $\mathbf{A}$, denoted by A_j, obeys a separate Euler–Lagrange equation.

In the presence of a charge density $\rho(\mathbf{r},t)$ and a current density $\mathbf{J}(\mathbf{r},t)$ the general Lagrangian density of classical electrodynamics can be written as

$$\mathscr{L} = \mathbf{J}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t) - \rho(\mathbf{r},t)\Phi(\mathbf{r},t) + \frac{\varepsilon_0}{2}E^2(\mathbf{r},t) - \frac{1}{2\mu_0}B^2(\mathbf{r},t)\,, \tag{1.5}$$

where $E^2 \equiv |\mathbf{E}|^2$ and $B^2 \equiv |\mathbf{B}|^2$ depend on Φ and $\mathbf{A}$ according to Eq. (1.1). When the Lagrangian density is varied with respect to Φ we obtain the Euler–Lagrange equation in Eq. (1.3), which can be written as Gauss' law

$$-\varepsilon_0\nabla\cdot\mathbf{E}(\mathbf{r},t) + \rho(\mathbf{r},t) = 0\,. \tag{1.6}$$

When we vary the Lagrangian density with respect to the components of $\mathbf{A}$, we find the Euler–Lagrange equations in Eq. (1.4). These can be reformulated as the Maxwell–Ampère law

$$\mathbf{J}(\mathbf{r},t) + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}(\mathbf{r},t) - \frac{1}{\mu_0}\nabla\times\mathbf{B}(\mathbf{r},t) = 0\,. \tag{1.7}$$

The relations in Eq. (1.1) and Eqs. (1.6) and (1.7) are equivalent to Maxwell's equations, as can be seen by taking the curl of **E** in Eq. (1.1):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \,. \tag{1.8}$$

The last Maxwell equation, $\nabla \cdot \mathbf{B} = 0$, is implicit in $\mathbf{B} = \nabla \times \mathbf{A}$ since the divergence of any curl vanishes.

It is well known that we have a gauge freedom in defining the potentials Φ and **A** that constitute the fields **E** and **B**. Since we are interested in radiation, it is convenient to adopt the Coulomb, or radiation, gauge

$$\nabla \cdot \mathbf{A} = 0 \quad \text{and} \quad \Phi = 0 \,. \tag{1.9}$$

In addition to the gauge choice, in this chapter we consider only the vacuum solutions of the electromagnetic fields:

$$\rho = 0 \quad \text{and} \quad \mathbf{J} = 0 \,. \tag{1.10}$$

When we now write Eq. (1.7) in terms of the potentials, we obtain the homogeneous wave equation for **A**

$$\nabla^2 \mathbf{A} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \,. \tag{1.11}$$

The classical solutions to this equation can be written as

$$\mathbf{A}(\mathbf{r}, t) = \sum_\lambda \int \frac{d\mathbf{k}}{\sqrt{\varepsilon_0}} \frac{A_\lambda(\mathbf{k}) \boldsymbol{\epsilon}_\lambda(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_\mathbf{k} t}}{\sqrt{(2\pi)^3 2\omega_\mathbf{k}}} + \text{c.c.}, \tag{1.12}$$

where $A_\lambda(\mathbf{k})$ denotes the amplitude of the mode with wave vector **k** and polarization λ, and c.c. denotes the complex conjugate. The vector $\boldsymbol{\epsilon}_\lambda$ gives the direction of the polarization, which we will discuss in Section 1.3. The dispersion relation for the free field is given by

$$|\mathbf{k}|^2 - \varepsilon_0 \mu_0 \, \omega_\mathbf{k}^2 \equiv k^2 - \frac{\omega_\mathbf{k}^2}{c^2} = 0 \,, \tag{1.13}$$

where c is the phase velocity of the wave with frequency $\omega_\mathbf{k}$. Any well-behaved potential $\mathbf{A}(\mathbf{r}, t)$ that can be expressed as a superposition of Fourier components is a solution to the wave equation. This is exemplified by the fact that we can see different shapes, colours, etc., rather than just uniform plane waves.

Finally, the Lagrangian density can be used to find the Hamiltonian density of the field. To this end, we define the canonical momenta of Φ and **A** as

$$\Pi_\Phi \equiv \frac{\delta \mathscr{L}}{\delta \dot{\Phi}} \quad \text{and} \quad \Pi_\mathbf{A} \equiv \frac{\delta \mathscr{L}}{\delta \dot{\mathbf{A}}} \,. \tag{1.14}$$

We can now take the Legendre transform of the Lagrangian density with respect to the dynamical variables $\dot{\Phi}$ and $\dot{\mathbf{A}}$ to obtain the Hamiltonian density of the free electromagnetic field

$$\mathcal{H}(\Phi, \mathbf{\Pi}_\Phi; \mathbf{A}, \mathbf{\Pi}_\mathbf{A}) = \mathbf{\Pi}_\Phi \dot{\Phi} + \mathbf{\Pi}_\mathbf{A} \cdot \dot{\mathbf{A}} - \mathscr{L} \,. \tag{1.15}$$

In the Coulomb gauge, the canonical momenta are

$$\Pi_\Phi = 0 \quad \text{and} \quad \Pi_\mathbf{A} = \varepsilon_0 \dot{\mathbf{A}} \,. \tag{1.16}$$

This leads to the Hamiltonian density for the free field

$$\mathcal{H} = \Pi_\mathbf{A} \cdot \dot{\mathbf{A}} - \mathscr{L}|_{\rho=\mathbf{J}=0} = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \,. \tag{1.17}$$

We now have all the necessary ingredients to proceed with the quantization of the electromagnetic field.

Exercise 1.1: Derive the homogeneous wave equation in Eq. (1.11) and show that the solutions are given by Eq. (1.12).

1.2 Quantization of the electromagnetic field

We are now ready to quantize the classical electromagnetic field. First, we have to decide which of the fields **A**, **E** (or **B**) we wish to quantize. In later chapters, we discuss the coupling between light and matter, and it is most convenient to express that coupling in terms of the vector potential **A**. We therefore apply the quantization procedure to **A**, rather than to **E**. In the quantization procedure we have to ensure that the quantum fields obey Maxwell's equations in the classical limit, and this leads to the introduction of a modified Dirac delta function. After the formal quantization, we explore the properties of the mode functions and the mode operators that result from the quantization procedure, and establish a fundamental relationship between them. We then construct eigenstates of the Hamiltonian, and define the discrete, physical modes. This leads to the concept of the photon. The final part of this section is devoted to the construction of the quantum mechanical field observables associated with single modes.

1.2.1 Field quantization

We denote the difference between classical and quantum mechanical observables by writing the latter with a hat. In the quantum theory of light, **A** and $\mathbf{\Pi}_\mathbf{A}$ then become operators satisfying the equal-time commutation relations. In index notation these are written as

$$\left[\hat{A}^j(\mathbf{r}, t), \hat{A}^k(\mathbf{r}', t)\right] = \left[\hat{\Pi}^j_\mathbf{A}(\mathbf{r}, t), \hat{\Pi}^k_\mathbf{A}(\mathbf{r}', t)\right] = 0 \,. \tag{1.18}$$

The field consists of four variables: three from **A** and one from Φ. We again work in the Coulomb gauge, where $\Phi = 0$ and $\nabla \cdot \mathbf{A} = 0$ ensures that we end up with only two dynamical variables.

Standard canonical quantization prescribes that, in addition to Eq. (1.18), we impose the following commutation relation:

$$\left[\hat{A}_i(\mathbf{r},t), \hat{\Pi}^j_{\mathbf{A}}(\mathbf{r}',t)\right] = i\hbar\delta_{ij}\,\delta^3(\mathbf{r}-\mathbf{r}')\,, \tag{1.19}$$

where we must remember the difference between upper and lower indices, $A_j = -A^j$, because electrodynamics is, at heart, a relativistic theory. Unfortunately, given that in the Coulomb gauge $\Pi^k_{\mathbf{A}} \propto E^k$, this commutation relation is not compatible with Gauss' law in vacuum: $\nabla \cdot \mathbf{E} = 0$. If we take the divergence with respect to the variable $\mathbf{r}'$ on both sides of Eq. (1.19), the left-hand side will be zero, but the divergence of the delta function does not vanish. We therefore have to modify the delta function such that its divergence does vanish. For the ordinary Dirac delta function we use the following definition:

$$\delta_{ij}\delta^3(\mathbf{r}-\mathbf{r}') \equiv \int \frac{d\mathbf{k}}{(2\pi)^3}\,\delta_{ij}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,. \tag{1.20}$$

We have included the Kronecker delta δ_{ij}, because after the redefinition of the delta function the internal degree of freedom j and the external degree of freedom $\mathbf{r}$ may no longer be independent (in fact, they will not be). Taking the divergence of Eq. (1.20) with respect to $\mathbf{r}$ yields

$$\sum_i \partial_i\,\delta_{ij}\,\delta^3(\mathbf{r}-\mathbf{r}') = i\int \frac{d\mathbf{k}}{(2\pi)^3}\,k_j e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,. \tag{1.21}$$

Therefore, we have to subtract something like this from the redefined delta function. We write

$$\Delta_{ij}(\mathbf{r}-\mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3}\,\delta_{ij}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - i\int \frac{d\mathbf{k}}{(2\pi)^3}\,\alpha_i k_j e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,, \tag{1.22}$$

and we want to find α_i such that $\partial_i\Delta_{ij}(\mathbf{r}-\mathbf{r}') = 0$:

$$\begin{aligned}
\sum_i \partial_i\Delta_{ij}(\mathbf{r}-\mathbf{r}') &= \sum_i \partial_i \int \frac{d\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left(\delta_{ij} - ik_j\alpha_i\right) \\
&= \sum_i \int \frac{d\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left[ik_i\delta_{ij} - (ik_j)(ik_i)\alpha_i\right] \\
&= 0\,.
\end{aligned} \tag{1.23}$$

We therefore have that

$$\sum_i \left(ik_i\delta_{ij} + k_ik_j\alpha_i\right) = 0\,, \quad \text{or} \quad \alpha_i = -i\frac{k_i}{|\mathbf{k}|^2}\,, \tag{1.24}$$

and the 'transverse' delta function $\Delta_{ij}(\mathbf{r}-\mathbf{r}')$ becomes

$$\Delta_{ij}(\mathbf{r}-\mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right) . \tag{1.25}$$

Using this modified delta function, we can complete the quantization procedure by imposing the equal-time canonical commutation relation

$$\left[\hat{A}_i(\mathbf{r},t), \hat{\Pi}^j_{\mathbf{A}}(\mathbf{r}',t)\right] = i\hbar\, \Delta_{ij}(\mathbf{r}-\mathbf{r}') . \tag{1.26}$$

That this leads to the correct covariant Hamiltonian and momentum is shown, for example, in Bjorken and Drell (1965). We can now write the three space components of the quantum field as

$$\hat{A}_j(\mathbf{r},t) = \sum_{\lambda=1}^{2} \int d\mathbf{k} \sqrt{\frac{\hbar}{\varepsilon_0}} \left[\epsilon_{\lambda j}(\mathbf{k})\hat{a}_\lambda(\mathbf{k})u(\mathbf{k};\mathbf{r},t) + \epsilon^*_{\lambda j}(\mathbf{k})\hat{a}^\dagger_\lambda(\mathbf{k})u^*(\mathbf{k};\mathbf{r},t)\right], \tag{1.27}$$

where the $u(\mathbf{k};\mathbf{r},t)$ are mode functions that are themselves solutions to the wave equation in Eq. (1.11), and λ again indicates the polarization of the electromagnetic field. The classical amplitudes $A_\lambda(\mathbf{k})$ are replaced by the operators $\hat{a}_\lambda(\mathbf{k})$, and $\hat{\mathbf{A}}$ is now a *quantum* field. Note that $\hat{\mathbf{A}}$ is now an operator, and unlike its classical counterpart does not directly represent a particular vector potential. Specific quantum mechanical vector potentials are represented by quantum states.

The equal-time commutation relation in Eq. (1.26) determines the commutation relation for $\hat{a}_\lambda(\mathbf{k})$ and $\hat{a}^\dagger_\lambda(\mathbf{k})$, given the mode functions $u(\mathbf{k};\mathbf{r},t)$ and $u^*(\mathbf{k},\mathbf{r},t)$. From Eq. (1.12) we can read off the plane-wave solutions with continuum normalization:

$$u(\mathbf{k};\mathbf{r},t) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}}t}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} , \tag{1.28}$$

where $\mathbf{k}$ is the wave vector of a wave with frequency $\omega_{\mathbf{k}}$. Plane waves are of constant intensity throughout space and time, and are therefore unphysical. However, they are mathematically very convenient. When the mode functions are the plane waves defined in Eq. (1.28), we find explicitly that

$$\left[\hat{a}_\lambda(\mathbf{k}), \hat{a}^\dagger_{\lambda'}(\mathbf{k}')\right] = \delta_{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}') , \tag{1.29}$$

and

$$\left[\hat{a}_\lambda(\mathbf{k}), \hat{a}_{\lambda'}(\mathbf{k}')\right] = \left[\hat{a}^\dagger_\lambda(\mathbf{k}), \hat{a}^\dagger_{\lambda'}(\mathbf{k}')\right] = 0 . \tag{1.30}$$

The operator $\hat{a}_\lambda(\mathbf{k})$ and its Hermitian conjugate are the 'mode operators' of the quantized electromagnetic field. In the next section we will see that any operators that obey these commutation relations are good mode operators.

We are now done with the quantization of the classical electromagnetic field, and the remainder of this section is devoted to the exploration of the direct consequences of this procedure.

Exercise 1.2: Derive the commutation relations in Eqs. (1.29) and (1.30).

1.2.2 Mode functions and mode operators

We will now discuss some of the fundamental properties of the mode functions $u(\mathbf{k};\mathbf{r},t)$ and $u^*(\mathbf{k};\mathbf{r},t)$, and the mode operators $\hat{a}_\lambda(\mathbf{k})$ and $\hat{a}_\lambda^\dagger(\mathbf{k})$. In order to study the mode functions of the field (its 'shape', if you like), we must first define a scalar product that allows us to talk about orthogonal mode functions. This is given by the 'time-independent scalar product'

$$(\phi,\psi) \equiv i\int d\mathbf{r}\,\phi^*\overleftrightarrow{\partial}_t\psi = i\int d\mathbf{r}\,\left[\phi^*(\partial_t\psi) - (\partial_t\phi^*)\psi\right] . \tag{1.31}$$

From the general structure of the scalar product in Eq. (1.31) we see that

$$(\phi,\psi)^* = (\psi,\phi) \quad \text{and} \quad (\phi^*,\psi^*) = -(\psi,\phi) . \tag{1.32}$$

This scalar product finds its origin in the continuity equation of the field, which determines the conserved currents (see Bjorken and Drell, 1965). It is therefore time-independent. The completeness relation of the mode functions $u(\mathbf{k};\mathbf{r},t)$ is then derived as follows: consider a function $f(\mathbf{r},t)$ that is a superposition of different mode functions

$$f = \int d\mathbf{k}\,\left[\alpha(\mathbf{k})u(\mathbf{k}) + \beta(\mathbf{k})u^*(\mathbf{k})\right], \tag{1.33}$$

where we have suppressed the dependence on $\mathbf{r}$ and t in $f(\mathbf{r},t)$ and $u(\mathbf{k};\mathbf{r},t)$ for notational brevity. Using the orthogonality of the mode functions defined by the scalar product, we can write the coefficients $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$ as

$$\alpha(\mathbf{k}) = (u(\mathbf{k}),f) \quad \text{and} \quad \beta(\mathbf{k}) = -(u^*(\mathbf{k}),f) . \tag{1.34}$$

This leads to an expression for f

$$f = \int d\mathbf{k}\,\left[(u(\mathbf{k}),f)\,u(\mathbf{k}) - (u^*(\mathbf{k}),f)\,u^*(\mathbf{k})\right]. \tag{1.35}$$

For a second superposition of mode functions g the scalar product (g,f) can be written as

$$(g,f) = \int d\mathbf{k}\,\left[(g,u(\mathbf{k}))(u(\mathbf{k}),f) - (g,u^*(\mathbf{k}))(u^*(\mathbf{k}),f)\right]. \tag{1.36}$$

This constitutes the 'completeness relation' for the mode functions $u(\mathbf{k};\mathbf{r},t)$, and it holds only if the mode functions are, in fact, complete.

Using the definition of the time-independent scalar product, we can show that plane-wave solutions are orthonormal:

$$\begin{aligned}
(u_{\mathbf{k}},u_{\mathbf{k}'}) &\equiv i\int d\mathbf{r}\,\frac{e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_{\mathbf{k}}t}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}}\overleftrightarrow{\partial}_t\frac{e^{i\mathbf{k}'\cdot\mathbf{r}-i\omega_{\mathbf{k}'}t}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}'}}} \\
&= \int\frac{d\mathbf{r}}{(2\pi)^3}\,\frac{(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}+i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} \\
&= \delta^3(\mathbf{k}-\mathbf{k}') .
\end{aligned} \tag{1.37}$$

We also find by direct evaluation that $(u_{\mathbf{k}}, u^*_{\mathbf{k}'}) = 0$. This can be understood physically as the orthogonality of waves moving forward in time, and waves moving backwards in time and in opposite directions. The plane waves therefore form a complete orthonormal set of mode functions.

We can define a new set of mode functions $v(\boldsymbol{\kappa}; \mathbf{r}, t)$, which are a linear combination of plane waves

$$v(\boldsymbol{\kappa}; \mathbf{r}, t) = \int d\mathbf{k}\, V(\boldsymbol{\kappa}, \mathbf{k}) u(\mathbf{k}; \mathbf{r}, t) = \int d\mathbf{k}\, V(\boldsymbol{\kappa}, \mathbf{k}) \frac{e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} . \tag{1.38}$$

Two symbols are needed for wave vectors here, namely $\mathbf{k}$ and $\boldsymbol{\kappa}$. We emphasize that we will normally reserve $\mathbf{k}$ for describing wave vectors. When $V(\boldsymbol{\kappa}, \mathbf{k})$ is unitary, the new mode functions $v(\boldsymbol{\kappa}; \mathbf{r}, t)$ are also orthonormal. When we express $\hat{\mathbf{A}}$ in terms of the new mode functions, we should also change the operators $\hat{a}_\lambda(\mathbf{k})$ to $\hat{b}_{\lambda'}(\boldsymbol{\kappa})$, since the mode operators depend on $\mathbf{k}$ and will generally change due to the transformation $V(\boldsymbol{\kappa}, \mathbf{k})$. The field operator then becomes

$$\hat{A}_j(\mathbf{r}, t) = \sum_{\lambda'=1}^{2} \int d\boldsymbol{\kappa} \sqrt{\frac{\hbar}{\varepsilon_0}} \left[\epsilon_{\lambda' j}(\boldsymbol{\kappa})\, \hat{b}_{\lambda'}(\boldsymbol{\kappa}) v(\boldsymbol{\kappa}; \mathbf{r}, t) + \epsilon^*_{\lambda' j}(\boldsymbol{\kappa})\, \hat{b}^\dagger_{\lambda'}(\boldsymbol{\kappa}) v^*(\boldsymbol{\kappa}; \mathbf{r}, t) \right] . \tag{1.39}$$

Note that here we have also included a possible change in the polarizarion degree of freedom λ', which can be incorporated straightforwardly in the time-independent scalar product.

Exercise 1.3: Prove the orthonormality of $v(\boldsymbol{\kappa}; \mathbf{r}, t)$ if V is unitary.

We next explore the precise relationship between mode functions and mode operators. The mode operators $\hat{a}_\lambda(\mathbf{k})$ and $\hat{a}^\dagger_\lambda(\mathbf{k})$ are related to the mode functions $u(\mathbf{k}; \mathbf{r}, t)$ and $u^*(\mathbf{k}; \mathbf{r}, t)$ via the time-independent scalar product

$$\begin{aligned} \hat{a}_{\lambda'}(\mathbf{k}) &\equiv \sqrt{\frac{\varepsilon_0}{\hbar}} \left(u(\mathbf{k}) \boldsymbol{\epsilon}_{\lambda'}, \hat{\mathbf{A}} \right) \\ &= i \sqrt{\frac{\varepsilon_0}{\hbar}} \int d\mathbf{r}\, u^*(\mathbf{k}; \mathbf{r}, t) \overleftrightarrow{\partial}_t \boldsymbol{\epsilon}^*_{\lambda'}(\mathbf{k}) \cdot \hat{\mathbf{A}}(\mathbf{r}, t) . \end{aligned} \tag{1.40}$$

We can then extract the operator $\hat{b}_{\lambda'}(\boldsymbol{\kappa})$, associated with the mode function $v(\boldsymbol{\kappa}; \mathbf{r}, t)$, using the procedure

$$\hat{b}_{\lambda'}(\boldsymbol{\kappa}) \equiv \sqrt{\frac{\varepsilon_0}{\hbar}} \left(v(\boldsymbol{\kappa}) \boldsymbol{\epsilon}_{\lambda'}, \hat{\mathbf{A}} \right) = i \sqrt{\frac{\varepsilon_0}{\hbar}} \int d\mathbf{r}\, v^*(\boldsymbol{\kappa}; \mathbf{r}, t) \overleftrightarrow{\partial}_t \boldsymbol{\epsilon}^*_{\lambda'}(\boldsymbol{\kappa}) \cdot \hat{\mathbf{A}}(\mathbf{r}, t) . \tag{1.41}$$

This is a *definition* of the operator $\hat{b}_{\lambda'}(\boldsymbol{\kappa})$, and is completely determined by the mode function $v(\boldsymbol{\kappa})$ and polarization vector $\boldsymbol{\epsilon}_{\lambda'}(\boldsymbol{\kappa})$. Now suppose that we have an expression for $\hat{\mathbf{A}}(\mathbf{r}, t)$ in terms of mode functions $u(\mathbf{k})$, mode operators $\hat{a}_\lambda(\mathbf{k})$, and polarization vectors $\boldsymbol{\epsilon}_\lambda(\mathbf{k})$ given in Eq. (1.27). The mode operator $\hat{b}_{\lambda'}(\boldsymbol{\kappa})$ then becomes

$$\hat{b}_{\lambda'}(\boldsymbol{\kappa}) = \sum_\lambda \int d\mathbf{k} \left[\boldsymbol{\epsilon}^*_{\lambda'}(\boldsymbol{\kappa}) \cdot \boldsymbol{\epsilon}_\lambda(\mathbf{k}) \,(v, u)\, \hat{a}_\lambda(\mathbf{k}) + \boldsymbol{\epsilon}^*_{\lambda'}(\boldsymbol{\kappa}) \cdot \boldsymbol{\epsilon}^*_\lambda(\mathbf{k}) \left(v, u^*\right) \hat{a}^\dagger_\lambda(\mathbf{k}) \right] , \tag{1.42}$$

where we express the mode operator $\hat{b}$ in terms of mode operators $\hat{a}$ and $\hat{a}^\dagger$. The spatial integration in the scalar products (v, u) and (v, u^*) must be evaluated before the integration over $\mathbf{k}$ in order to make the scalar product of the two polarization vectors $\boldsymbol{\epsilon}_{\lambda'}(\boldsymbol{\kappa}) \cdot \boldsymbol{\epsilon}_\lambda(\mathbf{k})$ definite. This demonstrates that the mode *operators* have a notion of orthogonality that is directly inherited from the orthogonality of the mode *functions*. Up to addition of a complex constant, Eq. (1.42) is the most general linear transformation of the mode operators, and is called the 'Bogoliubov transformation'. In principle it can mix the mode operators with their adjoints when the scalar product (v, u^*) is non-zero.

Exercise 1.4: Using Eq. (1.42), show that

$$\left[\hat{b}_\lambda(\boldsymbol{\kappa}), \hat{b}^\dagger_{\lambda'}(\boldsymbol{\kappa}')\right] = \delta_{\lambda\lambda'}\, \delta^3(\boldsymbol{\kappa} - \boldsymbol{\kappa}')\,, \tag{1.43}$$

and

$$\left[\hat{b}_\lambda(\boldsymbol{\kappa}), \hat{b}_{\lambda'}(\boldsymbol{\kappa}')\right] = \left[\hat{b}^\dagger_\lambda(\boldsymbol{\kappa}), \hat{b}^\dagger_{\lambda'}(\boldsymbol{\kappa}')\right] = 0 \tag{1.44}$$

These are the expected commutation relations for the mode operators.

1.2.3 Photons as excitations of the electromagnetic field

The revolutionary aspect of the quantum mechanical description of the electromagnetic field is the notion that the field can deliver its energy only in discrete amounts. This leads to the concept of the 'photon'. In order to derive this from the quantum theory, we first consider the Hamiltonian and momentum operators for the quantum field. We then construct energy eigenstates, and regularize them to obtain well-behaved physical states of the electromagnetic field.

From the quantum mechanical version of Eq. (1.17) we can formally derive the Hamiltonian operator $\mathcal{H}$ of the free field as

$$\begin{aligned}\mathcal{H} &= \sum_\lambda \int d\mathbf{k}\, \frac{\hbar\omega_\mathbf{k}}{2} \left[\hat{a}^\dagger_\lambda(\mathbf{k})\hat{a}_\lambda(\mathbf{k}) + \hat{a}_\lambda(\mathbf{k})\hat{a}^\dagger_\lambda(\mathbf{k})\right] \\ &\equiv \sum_\lambda \int d\mathbf{k}\, \mathcal{H}_\lambda(\mathbf{k})\,,\end{aligned} \tag{1.45}$$

where $\mathcal{H}_\lambda(\mathbf{k})$ will be called the 'single-mode Hamiltonian operator'. Similarly, the 'field momentum operator' is

$$\hat{\mathbf{P}} = \sum_\lambda \int d\mathbf{k}\, \frac{\hbar\mathbf{k}}{2} \left[\hat{a}^\dagger_\lambda(\mathbf{k})\hat{a}_\lambda(\mathbf{k}) + \hat{a}_\lambda(\mathbf{k})\hat{a}^\dagger_\lambda(\mathbf{k})\right]. \tag{1.46}$$

This operator is similar to $\mathcal{H}$, but $\omega_\mathbf{k}$ is replaced by $\mathbf{k}$. Therefore, the properties we derive for the Hamiltonian can easily be translated into properties for the momentum. The field momentum can be formally derived from Maxwell's stress tensor, but this is beyond the scope of this book.