# 1

# EQUIVALENT SINGLE-DEGREE-OF-FREEDOM SYSTEM AND FREE VIBRATION

The course on Mechanical Vibration is an important part of the Mechanical Engineering undergraduate curriculum. It is necessary for the development and the performance of many modern engineering products: automobiles, jet engines, rockets, bridges, electric motors, electric generators, and so on. Whenever a mechanical system contains storage elements for kinetic and potential energies, there will be vibration. The vibration of a mechanical system is a continual exchange between kinetic and potential energies. The vibration level is reduced by the presence of energy dissipation elements in the system. The problem of vibration is further accentuated because of the presence of time-varying external excitations, for example, the problem of resonance in a rotating machine, which is caused by the inevitable presence of rotor unbalance. There are many situations where the vibration is caused by internal excitation, which is dependent on the level of vibration. This type of vibration is known as self-excited oscillations, for example, the failure of the Tacoma suspension bridge (Billah and Scanlan, 1991) and the fluttering of an aircraft wing. This course deals with the characterization and the computation of the response of a mechanical system caused by time-varying excitations, which can be independent of or dependent on vibratory response. In general, the vibration level of a component of a machine has to be decreased to increase its useful life. As a result, the course also examines the methods used to reduce vibratory response. Further, this course also develops an input/output description of a dynamic system, which is useful for the design of a feedback control system in a future course in the curriculum.

The book starts with the definition of basic vibration elements and the vibration analysis of a single-degree-of-freedom (SDOF) system, which is the simplest lumped parameter mechanical system and contains one independent kinetic energy storage element (mass), one independent potential energy storage element (spring), and one independent energy dissipation element (damper). The analysis deals with natural vibration (without any external excitation) and forced response as well. The following types of external excitations are considered: constant, sinusoidal, periodic, and impulsive. In addition, an arbitrary nature of excitation is considered. Then, these analyses are presented for a complex lumped parameter mechanical system with multiple degrees of freedom (MDOF). The design of vibration absorbers is presented. Next, the vibration of a system with continuous distributions of mass, such as strings, longitudinal bars, torsional shafts, and beams, is presented. It is emphasized that the previous analyses of lumped parameter systems serve as building blocks for computation of the response of a continuous system that is governed by a partial differential equation. Last, the fundamentals of finite element analysis (FEA), which is widely used for vibration analysis of a real structure with a complex shape, are presented. This presentation again shows the application of concepts developed in the context of SDOF and MDOF systems to FEA.

In this chapter, we begin with a discussion of degrees of freedom and the basic elements of a vibratory mechanical system that are a kinetic energy storage element (mass), a potential energy storage element (spring), and an energy dissipation element (damper). Then, an SDOF system with many energy storage and dissipation elements, which are not independent, is considered. It is shown how an equivalent SDOF model with one equivalent mass, one equivalent spring, and one equivalent damper is constructed to facilitate the derivation of the differential equation of motion. Next, the differential equation of motion of an undamped SDOF spring–mass system is derived along with its solution to characterize its vibratory behavior. Then, the solution of the differential equation of motion of an SDOF spring–mass–damper system is obtained and the nature of the response is examined as a function of damping values. Three cases of damping levels, underdamped, critically damped, and overdamped, are treated in detail. Last, the concept of stability of an SDOF spring–mass–damper system is presented along with examples of self-excited oscillations found in practice.

## 1.1 DEGREES OF FREEDOM

*Degrees of freedom (**DOF**) are the number of independent coordinates that describe the position of a mechanical system at any instant of time.* For example, the system shown in Figure 1.1.1 has one degree of freedom $x$, which is the displacement of the mass $m_1$. In spite of the two masses $m_1$ and $m_2$ in Figure 1.1.2, this system has only one degree of freedom $x$ because both masses are connected by a rigid link, and the displacements of both masses are not independent. The system shown in Figure 1.1.3 has two degrees of freedom $x_1$ and $x_2$ because both masses $m_1$ and $m_2$ are connected by a flexible link or a spring, and the displacements of both masses are independent.

Next, consider rigid and flexible continuous cantilever beams as shown in Figures 1.1.4 and 1.1.5. The numbers of degrees of freedom for rigid and flexible beams are 0 and $\infty$, respectively. Each continuous beam can be visualized to contain an infinite number of point masses. These point masses are connected by rigid links for a rigid beam as shown in Figure 1.1.2, whereas they are connected by flexible links for a flexible beam as shown in Figure 1.1.3. Consequently, there is one degree of freedom associated with each of the point masses in a flexible beam.

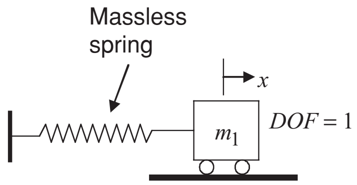


Figure 1.1.1 An SDOF system with a single mass

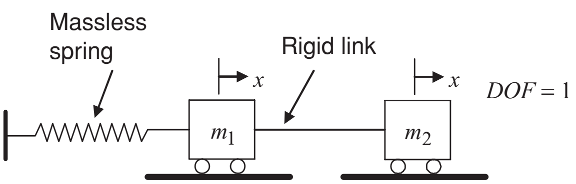


Figure 1.1.2 An SDOF system with two masses

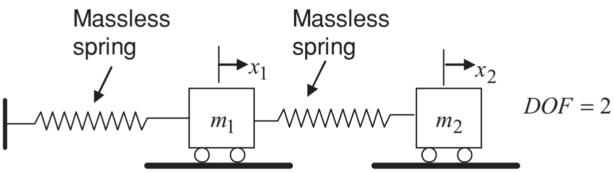


Figure 1.1.3 Two DOF systems with two masses

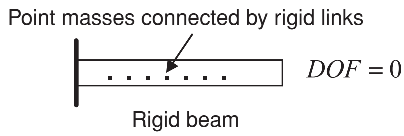


Figure 1.1.4 A rigid beam fixed at one end

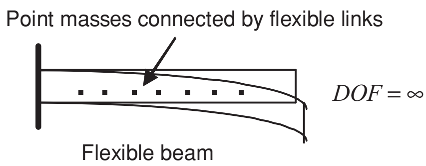


Figure 1.1.5 A flexible beam fixed at one end

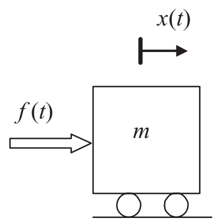


Figure 1.2.1 A mass in pure translation

## 1.2 ELEMENTS OF A VIBRATORY SYSTEM

There are three basic elements of a vibratory system: a kinetic energy storage element (mass), a potential energy storage element (spring), and an energy dissipation element (damper). The description of each of these three basic elements is as follows.

### *1.2.1 Mass and/or Mass-Moment of Inertia*

Newton's second law of motion and the expression of kinetic energy are presented for three types of motion: pure translational motion, pure rotational motion, and planar (combined translational and rotational) motion.

*Pure Translational Motion*

Consider the simple mass $m$ (Figure 1.2.1) which is acted upon by a force $f(t)$.

Applying Newton's second law of motion,

$$m\ddot{x} = f(t) \tag{1.2.1}$$

where

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \ddot{x} = \frac{d^2x}{dt^2} \tag{1.2.2a, b}$$

The energy of the mass is stored in the form of kinetic energy (KE):

$$\mathrm{KE} = \frac{1}{2}m\dot{x}^2 \tag{1.2.3}$$

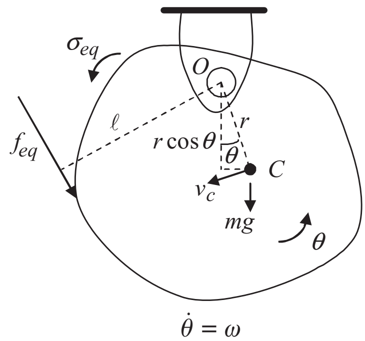


Figure 1.2.2 A mass in pure rotation

*Pure Rotational Motion*

Consider the mass $m$ (Figure 1.2.2) which is pinned at the point $O$, and acted upon by an equivalent external force $f_{eq}$ and an equivalent external moment $\sigma_{eq}$. This mass is undergoing a pure rotation about the point $O$, and Newton's second law of motion leads to

$$I_o\ddot{\theta} = -mgr\sin\theta + f_{eq}\ell + \sigma_{eq} \tag{1.2.4}$$

where $I_o$ is the mass-moment of inertia about the center of rotation $O$, $\theta$ is the angular displacement, and $\ell$ is the length of the perpendicular from the point $O$ to the line of force.

The KE of the rigid body is

$$\text{KE} = \frac{1}{2}I_o\dot{\theta}^2 \tag{1.2.5}$$

The potential energy (PE) of the rigid body is

$$\text{PE} = mg(r - r\cos\theta) \tag{1.2.6}$$

*Planar Motion (Combined Rotation and Translation) of a Rigid Body*

Consider the planar motion of a rigid body with mass $m$ and the mass-moment of inertia $I_c$ about the axis perpendicular to the plane of motion and passing through the center of mass $C$ (Figure 1.2.3). Forces $f_i, i = 1, 2, \ldots, n_f$, and moments $\sigma_i, i = 1, 2, \ldots, n_t$, are acting on this

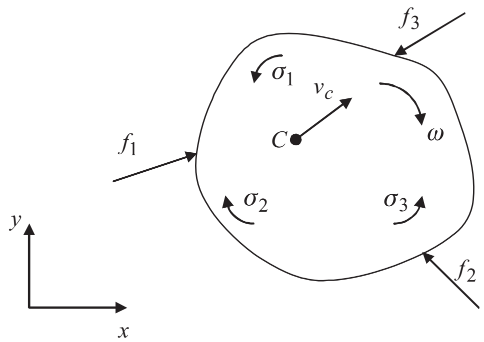


Figure 1.2.3 Planar motion of a rigid body

rigid body. Let $x_c$ and $y_c$ be $x$- and $y$- coordinates of the center of mass $C$ with respect to the fixed $x-y$ frame. Then, Newton's second law of motion for the translational part of motion is given by

$$m\ddot{x}_c = \sum_i f_{xi}(t) \tag{1.2.7}$$

$$m\ddot{y}_c = \sum_i f_{yi}(t) \tag{1.2.8}$$

where $f_{xi}$ and $f_{yi}$ are $x$- and $y$- components of the force $f_i$. Newton's second law of motion for the rotational part of motion is given by

$$I_c\ddot{\theta} = I_c\dot{\omega} = \sum_i \sigma_i(t) + \sum_i \sigma_{f_i}^c \tag{1.2.9}$$

where $\sigma_{f_i}^c$ is the moment of the force $f_i$ about the center of mass $C$. And, $\theta$ and $\omega$ are the angular position and the angular velocity of the rigid body, respectively. The KE of a rigid body in planar motion is given by

$$\mathrm{KE} = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2 \tag{1.2.10}$$

where $v_c$ is the magnitude of the linear velocity of the center of mass, that is,

$$v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 \tag{1.2.11}$$

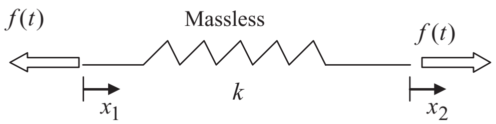


Figure 1.2.4 A massless spring in translation

*Special Case: Pure Rotation about a Fixed Point*

Note that the pure rotation of the rigid body (Figure 1.2.2) is a special planar motion for which

$$v_c = r\omega \tag{1.2.12}$$

and Equation 1.2.10 leads to

$$\text{KE} = \frac{1}{2}(mr^2 + I_c)\omega^2 \tag{1.2.13}$$

Using the parallel-axis theorem,

$$I_o = I_c + mr^2 \tag{1.2.14}$$

Therefore, Equation 1.2.5 is obtained for the case of a pure rotation about a fixed point.

### *1.2.2 Spring*

The spring constant or stiffness and the expression of PE are presented for two types of motion: pure translational motion and pure rotational motion.

*Pure Translational Motion*

Consider a massless spring, subjected to a force $f(t)$ on one end (Figure 1.2.4). Because the mass of the spring is assumed to be zero, the net force on the spring must be zero. As a result, there will be an equal and opposite force on the other end. The spring deflection is the difference between the displacements of both ends, that is,

$$\text{spring deflection} = x_2 - x_1 \tag{1.2.15}$$

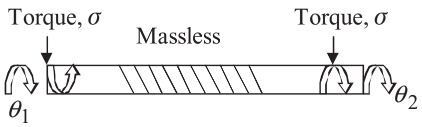


Figure 1.2.5 A massless spring in rotation

and the force is directly proportional to the spring deflection:

$$f(t) = k(x_2 - x_1) \tag{1.2.16}$$

where the proportionality constant $k$ is known as the spring constant or stiffness.

The PE of the spring is given by

$$\mathrm{PE} = \frac{1}{2}k(x_2 - x_1)^2 \tag{1.2.17}$$

It should be noted that the PE is independent of the sign (extension or compression) of the spring deflection, $x_2 - x_1$.

*Pure Rotational Motion*

Consider a massless torsional spring, subjected to a torque $\sigma(t)$ on one end (Figure 1.2.5). Because the mass of the spring is assumed to be zero, the net torque on the spring must be zero. As a result, there will be an equal and opposite torque on the other end. The spring deflection is the difference between angular displacements of both ends, that is,

$$\text{spring deflection} = \theta_2 - \theta_1 \tag{1.2.18}$$

and the torque is directly proportional to the spring deflection:

$$\sigma(t) = k_t(\theta_2 - \theta_1) \tag{1.2.19}$$

where the proportionality constant $k_t$ is known as the torsional spring constant or torsional stiffness.

The PE of the torsional spring is given by

$$\mathrm{PE} = \frac{1}{2}k_t(\theta_2 - \theta_1)^2 \tag{1.2.20}$$

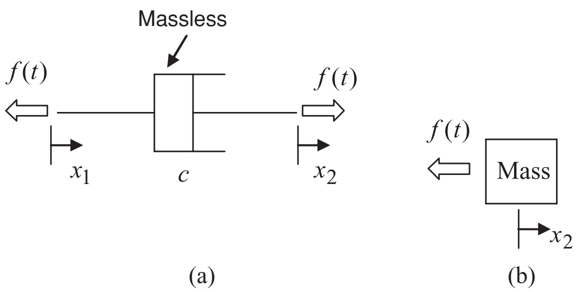


Figure 1.2.6 (a) A massless damper in translation; (b) A mass attached to the right end of the damper

It should be noted that the PE is independent of the sign of the spring deflection, $\theta_2 - \theta_1$.

### *1.2.3 Damper*

The damping constant and the expression of energy dissipation are presented for two types of motion: pure translational motion and pure rotational motion.

*Pure Translational Motion*

Consider a massless damper, subjected to force $f(t)$ on one end (Figure 1.2.6a). Because the mass of the damper is assumed to be zero, the net force on the damper must be zero. As a result, there will be an equal and opposite force on the other end, and the damper force is directly proportional to the difference of the velocities of both ends:

$$f(t) = c(\dot{x}_2(t) - \dot{x}_1(t)) \tag{1.2.21}$$

where the proportionality constant $c$ is known as the damping constant. The damper defined by Equation 1.2.21 is also known as the linear viscous damper.

If there is a mass attached to the damper at the right end (Figure 1.2.6b) with the displacement $x_2$, the work done on the mass