1
Introduction

A recurrent theme in this book is the concept of a game. There are essentially three kinds of games in logic. One is the Semantic Game, also called the Evaluation Game, where the *truth* of a given sentence in a given model is at issue. Another is the Model Existence Game, where the *consistency* in the sense of having a model, or equivalently in the sense of impossibility to derive a contradiction, is at issue. Finally there is the Ehrenfeucht–Fraïssé Game, where *separation* of a model from another by finding a property that is true in one given model but false in another is the goal. The three games are closely linked to each other and one can even say they are essentially variants of just one basic game. This basic game arises from our understanding of the quantifiers. The purpose of this book is to make this strategic aspect of logic perfectly transparent and to show that it underlies not only first-order logic but infinitary logic and logic with generalized quantifiers alike.

We call the close link between the three games the *Strategic Balance of Logic* (Figure 1.1). This balance is perfectly commutative, in the sense that winning strategies can be transferred from one game to another. This mere fact is testimony to the close connection between logic and games, or, thinking semantically, between games and models. This connection arises from the nature of quantifiers. Introducing infinite disjunctions and conjunctions does not upset the balance, barring some set-theoretic issues that may surface. In the last chapter of this book we consider generalized quantifiers and show that the Strategic Balance of Logic persists even in the presence of generalized quantifiers.

The purpose of this book is to present the Strategic Balance of Logic in all its glory.

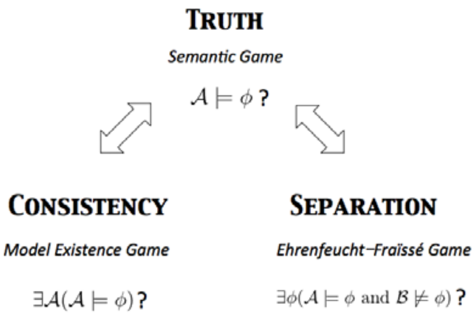

Figure 1.1 The Strategic Balance of Logic.

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Preliminaries and Notation

We use some elementary set theory in this book, mainly basic properties of countable and uncountable sets. We will occasionally use the concept of countable ordinal when we index some uncountable sets. There are many excellent books on elementary set theory. (See Section 2.7.) We give below a simplified account of some basic concepts, the barest outline necessary for this book.

We denote the set $\{0, 1, 2, \ldots\}$ of all natural numbers by $\mathbb{N}$, the set of rational numbers by $\mathbb{Q}$, and the set of all real numbers by $\mathbb{R}$. The power-set operation is written

$$\mathcal{P}(A) = \{B : B \subseteq A\}.$$

We use $A \setminus B$ to denote the set-theoretical difference of the sets A and B. If f is a function, $f''X$ is the set $\{f(x) : x \in X\}$ and $f^{-1}(X)$ is the set $\{x \in \mathrm{dom}(f) : f(x) \in X\}$. *Composition* of two functions f and g is denoted $g \circ f$ and defined by $(g \circ f)(x) = g(f(x))$. We often write fa for $f(a)$. The notation id_A is used for the *identity function* $A \to A$ which maps every element of A to itself, i.e. $id_A(a) = a$ for $a \in A$.

2.1 Finite Sequences

The concept of a finite (ordered) sequence

$$s = (a_0, \ldots, a_{n-1})$$

of elements of a given set A plays an important role in this book. Examples of finite sequences of elements of $\mathbb{N}$ are

$$(8, 3, 9, 67, 200, 0)$$
$$(8, 8, 8)$$

$$(24).$$

We can identify the sequence $s = (a_0, \ldots, a_{n-1})$ with the function

$$s' : \{0, \ldots, n-1\} \to A,$$

where

$$s'(i) = a_i.$$

The main property of finite sequences is: $(a_0, \ldots, a_{n-1}) = (b_0, \ldots, b_{m-1})$ if and only if $n = m$ and $a_i = b_i$ for all $i < n$. The number n is called the *length* of the sequence $s = (a_0, \ldots, a_{n-1})$ and is denoted $\mathrm{len}(s)$. A special case is the case $\mathrm{len}(s) = 0$. Then s is called the empty sequence. There is exactly one empty sequence and it is denoted by $\emptyset$.

The Cartesian product of two sets A and B is written

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

More generally

$$A_0 \times \ldots \times A_{n-1} = \{(a_0, \ldots, a_{n-1}) : a_i \in A_i \text{ for all } i < n\}.$$

$$A^n = A \times \ldots \times A \ (n \text{ times}).$$

According to this definition, $A^1 \neq A$. The former consists of sequences of length 1 of elements of A. Note that $A^0 = \{\emptyset\}$.

Finite Sets

A set A is *finite* if it is of the form $\{a_0, \ldots, a_{n-1}\}$ for some natural number n. This means that the set A has at most n elements. If A has exactly n elements we write $|A| = n$ and call $|A|$ the cardinality of A. A set which is not finite is *infinite*. Finite sets form a so-called *ideal*, which means that:

1. $\emptyset$ is finite.
2. If A and B are finite, then so is $A \cup B$.
3. If A is finite and $B \subseteq A$, then also B is finite.

Further useful properties of finite sets are:

1. If A and B are finite, then so is $A \times B$.
2. If A is finite, then so is $\mathcal{P}(A)$.

The *Axiom of Choice* says that for every set A of non-empty sets there is a function f such that $f(a) \in a$ for all $a \in A$. We shall use the Axiom of Choice freely without specifically mentioning it. It needs some practice in set theory to see how the axiom is used. Often an intuitively appealing argument involves a hidden use of it.

Lemma 2.1 *A set A is finite if and only if every injective $f : A \to A$ is a bijection.*

Proof Suppose A is finite and $f : A \to B$ is an injection with $B \subset A$ and $a \in A \setminus B$. Let $a_0 = a$ and $a_{n+1} = f(a_n)$. It is easy to see that $a_n \neq a_m$ whenever $n < m$, so we contradict the finiteness of A. On the other hand, if A is infinite, we can (by using the Axiom of Choice) pick a sequence b_n, $n \in \mathbb{N}$, of distinct elements from A. Then the function g which maps each b_n to b_{n+1} and is the identity elsewhere is an injective mapping from A into A but not a bijection. □

The set of all n-element subsets $\{a_0, \ldots, a_{n-1}\}$ of A is denoted by $[A]^n$.

2.2 Equipollence

Sets A and B are *equipollent*

$$A \sim B$$

if there is a bijection $f : A \to B$. Then $f^{-1} : B \to A$ is a bijection and

$$B \sim A$$

follows. The composition of two bijections is a bijection, whence

$$A \sim B \sim C \Longrightarrow A \sim C.$$

Thus $\sim$ divides sets into equivalence classes. Each equivalence class has a canonical representative (a cardinal number, see the Subsection "Cardinals" below) which is called the *cardinality* of (each of) the sets in the class. The cardinality of A is denoted by $|A|$ and accordingly $A \sim B$ is often written

$$|A| = |B|.$$

One of the basic properties of equipollence is that if

$$A \sim C, B \sim D \text{ and } A \cap B = C \cap D = \emptyset,$$

then

$$A \cup B \sim C \cup D.$$

Indeed, if $f : A \to C$ is a bijection and $g : B \to D$ is a bijection, then $f \cup g : A \cup B \to C \cup D$ is a bijection. If the assumption

$$A \cap B = C \cap D = \emptyset$$

is dropped, the conclusion fails, of course, as we can have $A \cap B = \emptyset$ and $C = D$. It is also interesting to note that even if $A \cap B = C \cap D = \emptyset$, the assumption $A \cup B \sim C \cup D$ does not imply $B \sim D$ even if $A \sim C$ is assumed: Let $A = \mathbb{N}, B = \emptyset, C = \{2n : n \in \mathbb{N}\}$, and $D = \{2n+1 : n \in \mathbb{N}\}$. However, for finite sets this holds: if $A \cup B$ is finite,

$$A \cup B \sim C \cup D,\ A \sim C,\ A \cap B = C \cap D = \emptyset$$

then

$$B \sim D.$$

We can interpret this as follows: the cancellation law holds for finite numbers but does not hold for cardinal numbers of infinite sets.

There are many interesting and non-trivial properties of equipollence that we cannot enter into here. For example the Schröder–Bernstein Theorem: If $A \sim B$ and $B \subseteq C \subseteq A$, then $A \sim C$. Here are some interesting consequences of the Axiom of Choice:

- For all A and B there is C such that $A \sim C \subseteq B$ or $B \sim C \subseteq A$.
- For all infinite A we have $A \sim A \times A$.

It is proved in set theory by means of the Axiom of Choice that $|A| \leq |B|$ holds in the above sense if and only if the cardinality $|A|$ of the set A is at most the cardinality $|B|$ of the set B. Thus the notation $|A| \leq |B|$ is very appropriate.

2.3 Countable sets

A set A which is empty or of the form $\{a_0, a_1, \ldots\}$, i.e. $\{a_n : n \in \mathbb{N}\}$, is called *countable*. A set which is not countable is called *uncountable*. The countable sets form an ideal just as the finite sets do. We now prove two important results about countability. Both are due to Georg Cantor:

Theorem 2.2 *If A and B are countable, then so is $A \times B$.*

Proof If either set is empty, the Cartesian product is empty. So let us assume the sets are both non-empty. Suppose $A = \{a_0, a_1, \ldots\}$ and $B = \{b_0, b_1, \ldots\}$. Let

$$c_n = \begin{cases} (a_i, b_j), & \text{if } n = 2^i 3^j \\ (a_0, b_0), & \text{otherwise.} \end{cases}$$

Now $A \times B = \{c_n : n \in \mathbb{N}\}$, whence $A \times B$ is countable. □

Theorem 2.3 *The union of a countable family of countable sets is countable.*

Proof The empty sets do not contribute anything to the union, so let us assume all the sets are non-empty. Suppose A_n is countable for each $n \in \mathbb{N}$, say, $A_n = \{a^n_m : m \in \mathbb{N}\}$ (we use here the Axiom of Choice to choose an enumeration for each A_n). Let $B = \bigcup_n A_n$. We want to represent B in the form $\{b_n : n \in \mathbb{N}\}$. If n is given, we consider two cases: If n is $2^i 3^j$ for some i and j, we let $b_n = a^i_j$. Otherwise we let $b_n = a^0_0$. □

Theorem 2.4 *The power-set of an infinite set is uncountable.*

Proof Suppose A is infinite and $\mathcal{P}(A) = \{b_n : n \in \mathbb{N}\}$. Since A is infinite, we can choose distinct elements $\{a_n : n \in \mathbb{N}\}$ from A. (This uses the Axiom of Choice. For an argument which avoids the Axiom of Choice see Exercise 2.14.) Let

$$B = \{a_n : a_n \notin b_n\}.$$

Since $B \subseteq A$, there is some n such that $B = b_n$. Is a_n an element of B or not? If it is, then $a_n \notin b_n$ which is a contradiction. So it is not. But then $a_n \in b_n = B$, again a contradiction. □

2.4 Ordinals

The ordinal numbers introduced by Cantor are a marvelous general theory of measuring the *potentially infinite*. They are intimately related to inductive definitions and occur therefore widely in logic. It is easiest to understand ordinals in the context of games, although this was not Cantor's way. Suppose we have a game with two players **I** and **II**. It does not matter what the game is, but it could be something like chess. If **II** can force a win in n moves we say that the game has *rank* n. Suppose then **II** cannot force a win in n moves for any n, but after she has seen the first move of **I**, she can fix a number n and say that she can force a win in n moves. This situation is clearly different from being able to say in advance what n is. So we invent a symbol ω for the rank of this game. In a clear sense ω is greater than each n but there does not seem to be any possible rank between all the finite numbers n and ω. We can think of ω as an infinite number. However, there is nothing metaphysical about the infiniteness of ω. It just has infinitely many predecessors. We can think of ω as a tree T_ω with a root and a separate branch of length n for each n above the root as in the tree on the left in Figure 2.1.

Figure 2.1 T_ω and $T_{\omega+1}$.

Suppose then **II** is not able to declare after the first move how many moves she needs to beat **II**, but she knows how to play her first move in such a way that after **I** has played his second move, she can declare that she can win in n moves. We say that the game has rank $\omega + 1$ and agree that this is greater than ω but there is no rank between them. We can think of $\omega + 1$ as the tree which has a root and then above the root the tree T_ω, as in the tree on the right in Figure 2.1. We can go on like this and define the ranks $\omega + n$ for all n.

Suppose now the rank of the game is not any of the above ranks $\omega + n$, but still **II** can make an interesting declaration: she says that after the first move of **I** she can declare a number m so that after m moves she declares another number n and then in n moves she can force a win. We would say that the rank of the game is $\omega + \omega$. We can continue in this way defining ranks of games that are always finite but potentially infinite. These ranks are what set theorists call ordinals.

We do not give an exact definition of the concept of an ordinal, because it would take us too far afield and there are excellent textbooks on the topic. Let us just note that the key properties of ordinals and their total order $<$ are:

1. Natural numbers are ordinals.
2. For every ordinal α there is an immediate successor $\alpha + 1$.
3. Every non-empty set of ordinals has a smallest element.
4. Every non-empty set of ordinals has a supremum (i.e. a smallest upper bound).

The supremum of the set $\{0, 1, 2, 3, \ldots\}$ of ordinals is denoted by ω. An ordinal is said to be *countable* if it has only countably many predecessors, otherwise *uncountable*. The supremum of all countable ordinals is denoted by ω_1. Here is a picture of the ordinal number "line":

$$0 < 1 < 2 < \ldots < \omega < \omega + 1 < \ldots < \alpha < \alpha + 1 < \ldots < \omega_1 < \ldots$$

Ordinals that have a last element, i.e. are of the form $\alpha + 1$, are called *successor* ordinals; the rest are *limit* ordinals, like ω and $\omega + \omega$.

Ordinals are often used to index elements of uncountable sets. For example, $\{a_\alpha : \alpha < \beta\}$ denotes a set whose elements have been indexed by the ordinal β, called the *length* of the sequence. The set of all such sequences of length β of elements of a given set A is denoted by A^β. The set of all sequences of length $< \beta$ of elements of a given set A is denoted by $A^{<\beta}$.

2.5 Cardinals

Historically cardinals (or more exactly cardinal numbers) are just representatives of equivalence classes of equipollence. Thus there is a cardinal number for countable sets, denoted $\aleph_0$, a cardinal number for the set of all reals, denoted $\mathfrak{c}$, and so on. There is some question as to what exactly are these cardinal numbers. The Axiom of Choice offers an easy answer, which is the prevailing one, as it says that every set can be well-ordered. Then we can let the cardinal number of a set be the order-type of the smallest well-order equipollent with the set. Equivalently, the cardinal number of a set is the smallest ordinal equipollent with the set. If we leave aside the Axiom of Choice, some sets need not have have a cardinal number. However, as is customary in current set theory, let us indeed assume the Axiom of Choice. Then every set has a cardinal number and the cardinal numbers are ordinals, hence well-ordered. The α^{th} infinite cardinal number is denoted $\aleph_\alpha$. Thus $\aleph_1$ is the next in order of magnitude from $\aleph_0$. The famous *Continuum Hypothesis* is the statement that $\aleph_1 = \mathfrak{c}$.

For every set A there exists (by the Axiom of Choice) an ordinal α such that the elements of A can be listed as $\{a_\beta : \beta < \alpha\}$. The smallest such α is called the *cardinal number*, or *cardinality*, of A and denoted by $|A|$. Thus certain ordinals are cardinal numbers of sets. Such ordinals are called *cardinals*. They are considered as canonical representatives of each equivalence class of equipollent sets. For example, all finite numbers are cardinals, as are ω and ω_1. The smallest cardinal such that the smaller infinite cardinals can be enumerated in increasing order as κ_β, $\beta < \alpha$, is denoted ω_α, or alternatively $\aleph_\alpha$. If

$\kappa = \aleph_\alpha$, then $\aleph_{\alpha+1}$ is denoted κ^+ and is called a *successor cardinal*. Cardinals that are not successor cardinals are called *limit cardinals*.

Arithmetic operations $\kappa + \lambda, \kappa \cdot \lambda, \kappa^\lambda$ for cardinals are defined as follows:

$$\kappa + \lambda = |\kappa \cup \lambda|, \ \kappa \cdot \lambda = |\kappa \times \lambda|.$$

Moreover, exponentiation κ^λ of cardinal numbers is defined as the cardinality of the set κ^λ of sequences of elements of κ of length λ. A certain amount of knowledge about the arithmetic of cardinal numbers in necessary in this book, especially in the later chapters, and Chapters 8 and 9 in particular.

The *cofinality* of an ordinal α is the smallest ordinal β for which there is a function $f : \beta \to \alpha$ such that (1) $\xi < \zeta < \beta$ implies $f(\xi) < f(\zeta)$, and (2) for all $\xi < \alpha$ there is some $\zeta < \beta$ such that $\xi < f(\zeta)$. We use $\mathrm{cf}(\alpha)$ to denote the cofinality of α. A cardinal κ is said to be *regular* if $\mathrm{cf}(\kappa) = \kappa$, and *singular* if $\mathrm{cf}(\kappa) < \kappa$. Successor cardinals are always regular. The smallest singular cardinal is $\aleph_\omega$.

The *Continuum Hypothesis* (CH) is the hypothesis $|\mathcal{P}(\mathbb{N})| = \aleph_1$. Neither it nor its negation can be derived from the usual *Zermelo–Fraenkel axioms* of set theory and therefore it (or its negation), like many other similar hypotheses, has to be explicitly mentioned as an assumption, when it is used.

2.6 Axiom of Choice

We have already mentioned the Axiom of Choice. There are so many equivalent formulations of this axiom that books have been written about it. The most notable formulation is the Well-Ordering Principle: every set is equipollent with an ordinal. The Axiom of Choice is sometimes debated because it brings arbitrariness or abstractness into mathematics, often with examples that can be justifiably called pathological, like the Banach–Tarski Paradox: The unit sphere in three-dimensional space can be split into five pieces so that if the pieces are rigidly moved and rotated they form two spheres, each of the original size. The trick is that the splitting exists only in the abstract world of mathematics and can never actually materialize in the physical world. Conclusion: infinite abstract objects do not obey the rules we are used to among finite concrete objects. This is like the situation with sub-atomic elementary particles, where counter-intuitive phenomena, such as entanglement, occur.

Because of the abstractness brought about by the Axiom of Choice it has received criticism and some authors always mention explicitly if they use it in their work. The main problem in working *without* the Axiom of Choice is