

INDEX

- $(Z \models_0 X)$, 383
 $(Z)^x$, *see also* sequence, coding
 $(Z \models_i X)$, 401
 $(Z \models_{\Sigma_i^q} X)$, $(Z \models_{\Pi_i^q} X)$, 385
 $(Z \models_{\Sigma_0^q} X)$, $(Z \models_{\Pi_0^q} X)$, 394
 $(Z \models_0^{\Sigma} X)$, $(Z \models_0^{\Sigma} X)$, 384
 $(Z \models X)$, 382
 $(\forall 2^i)$, $(\exists 2^i)$, 61
 $=, =^{\mathcal{M}}$, 19
 $A \iff B$, 10, 20
 $A \leftrightarrow B$, 9
 $A \supset B$, 9
 A^{τ} , 9
 $B \models A$, 21
 $B_F(i, \vec{x}, \vec{Y})$, 101
 $G_f(z, \vec{x}, \vec{Y})$, 101
 $Q_{R_1} \leq_{AC^0} Q_{R_2}$, 229
 $R_+(X, Y, Z)$, *see also* addition, 85
 $R_{\times}(X, Y, Z)$, *see also* multiplication, 85
 $S(X)$, *see also* successor function, 112
 $S(X, Y)$, 378
 $S_k(X, Y)$, 379
 $T_{\varphi}(\vec{m}, \vec{n})$, 374
 $U *_{\tau} V$, 231
 $X(t)$, 76
 $X + Y$, *see also* addition, 102
 $X[i, j]$, *see also* substring function, 366
 $X \div Y$, $\lfloor X/Y \rfloor$, *see also* division, 137
 $X \leq Y$, $X < Y$, 219
 $X \times Y$, *see also* multiplication, 136
 $X^{<x}$, $Cut(x, X)$, 139
 $Z \dot{-} Y$, 220
 $Z^{\lfloor x \rfloor}$, $Row(x, Z)$, 114
 $\Delta_0(\mathcal{L})$, 51
 $FLA_{\mathcal{F}}$, 367
 $Fla^{\Sigma}(X)$, $Fla^{\Pi}(X)$, 368
 \mathcal{L}_A , 18
 \mathcal{L}_{FO} , 74
 $\Phi \models A$, 20
 $\Phi \vdash A$, 33
 $Fla_{\Phi}^{\Pi}(X)$, $Fla_{\Phi}^{\Sigma}(X)$, 396
 $\Phi \models A$, $\models A$, 10
 $\Pi_i^B(\mathcal{L})$, $\Sigma_i^B(\mathcal{L})$, 82
 $Prf_{\mathcal{F}}^{\Sigma}(\pi, X)$, $Prf_{\mathcal{F}}^{\Pi}(\pi, X)$, 367
 $\Sigma_j^B(V^i)$, 403
 $\Sigma_1(\mathcal{L})$, 51
 Σ_1^1 , 82
 $\Sigma_0^B(\Phi)$, 109
 Σ_0^P , 130
 Th_k , 428
 $bin(X)$, 85
 $\mathcal{M} \models A$, 19
 $\mathcal{M} \models A[\sigma]$, 20
 $\mathcal{M} \models \Phi[\sigma]$, 20
 $\mathcal{T}_1 \subset_{cons} \mathcal{T}_2$, 201
 $=_{syn}$, 10
 $\exists X \leq T, \forall X \leq T$, 219
 $\exists \vec{x}$, 42
 $\exists x \leq t, \forall x \leq t$, 42
 $\exists!$, 50
 $\lfloor x/y \rfloor$, 60
 \longrightarrow (empty sequent), 10
 $\models A$, 20
 $parity(X)$, 118
 $seq(x, Z)$, $(Z)^x$, 115
 $\underline{\mathbb{N}}_2$, 79
 \top , \perp , 9
 $val(t)$, 166
 \emptyset , *see also* empty set, 112
 $\varphi(X_0[\])$, 369
 $\varphi_{\mathcal{F}}(y, \pi, X, Y)$, $t_{\mathcal{F}}(\pi, X)$, 367
 $\varphi_{FLA}(y, X, Y)$, t_{FLA} , 368
 φ^b , 260
 φ^{rec} , *see also* **BIT-REC**
 \widehat{A} , 382
 $\{x\}$, *see also* **POW2**
 $\{x\}$, **POW2**(x), 220
 $f_{\varphi(z), t}(\vec{x})$, 55

- p -simulation, 161
 \mathbf{G}_{i+1}^* p -simulates \mathbf{G}_i w.r.t. $\Sigma_i^q \cup \Pi_i^q$, 179
 \mathbf{G}_i p -simulates \mathbf{G}_{i+1}^* , 182
 \mathbf{G}_0^* p -simulates \mathbf{G}_0 w.r.t. prenex Σ_1^q , 180
 $t(s/x)$, $A(s/x)$, 21
 $t < u$, 40
 $t^{\mathcal{M}}[\sigma]$, 20
 $x \dot{=} y$, 60
 $|X|$, length function for string, 76
 $|\vec{T}|$, 81
 $|x|$ in \mathbf{ID}_0 , 64
 $\#$, 70, 256
 Δ_i^B formula, 212
 Δ_0 , 42
 Δ_0^N , 67
 $F_{\varphi(z), t}$, $f_{\varphi(z), t}$, 125
 \mathcal{L}_A^2 , 76
 $\mathcal{M}^\#$, 258
 \mathcal{N}^b , 259
 $PRF_{\mathcal{F}}$, 367
 Π_i^p , 451, 453
 RFN_{ePK} , 402
 Φ - $RFN_{\mathcal{F}}$, 396
 Σ_i^{\log} , 84
 Σ_i^{\ln} , 66, 452
 Σ_i^b , 70
 Σ_i^p , 65, 451, 453
 Σ_i^q , Π_i^q , 174
 Σ_1 , 42
 Σ_1^p , 156
 f^* , F^* , *see also* aggregate function
 $g\Sigma_i^B$, 142
 \underline{n} , 40
 pd , 125
 Σ_i^q - $RFN_{\mathcal{F}}(\pi_0, X_0, Z)[k]$, 407
 $\varphi(\vec{x}, \vec{X})[\vec{m}; \vec{n}]$, 166
 $\|\varphi(\vec{x}, \vec{X})\|$, 166
 $\psi^\#$, 261
 $qdepth$, 373
 $single$ - Σ_1^B , 115
 \mathbb{N} , 20
 AC hierarchy, *see also* NC hierarchy
 AC^0 , 73, 75, 83, 84, 95
 $AC^0/poly$, 75
 FAC^0 closed under AC^0 -reduction, 270
characteristic functions of, 117
closure, *see also* closure
reduction, *see also* reduction
theories for, *see also* V^0
 $AC^0(2)$, 308
characterized by 2-BNR, 308
theories for, *see also* $V^0(2)$, $VAC^0(2)V$
 $AC^0(6)$, 318
characterized by 4-BNR and 3-BNR, 318
theories for, *see also* $VAC^0(6)V$, $V^0(m)$
 AC^0 function, *see also* FAC^0
 AC^0 -Frege, 164
 AC^0 -ITERATION, 230
 $AC^0(m)$, 303, 313
definition of, 456
theories for, *see also* $V^0(m)$
 AC^0 -PLS, 230
 AC^1 , 339
 ACC , 303, 313
definition of, 456
theories for, *see also* $VACC$
 AC^k , 319, 320
definition of, 455
active sequent, 27
in LK^2+IND , 152
adding n strings in TC^0 , 294
addition
 $R_+(X, Y, Z)$, 85
by divide-and-conquer, 330
carry-lookahead adder, 85
string function $X + Y$, 102, 112
Adequacy Theorem for $LK-ID_0$, 46
aggregate function, 201, 205, 273, 281
in Elimination Theorem, 276, 291
Aggregate Function Theorem, 208
algorithms
for HornSat, 224
 $A\text{LogTime}$, *see also* NC^1 , 321, 455
 $D\text{LogTime}$, 449
alternating Turing machine, *see also* ATM
Anchored Completeness Theorem
for LK , 30
for LK with $=$, 33
for LK^2+IND , 151
for PK , 14
anchored proof, 14
 LK , 30, 33
 LK^2 , 89
 LK subformula property, 33
with Φ - IND rule, 151
antecedent, 10
arithmetical functions, 107
 $ASpace$ -Alt, 320, 453, 455
 $ASpace$ -Time, 320, 453, 455
 $A\text{Time}$, 453
 $A\text{Time}$ -Alt, 453, 455
ATM, 320, 452
atom, 9

- atomic formula, 9
 auxiliary formula, 11, 22
 auxiliary formulas, 175
 axiom, 39
 nonlogical, 13
 axiom scheme
 number vs. string, 97
 bit recursion, *see also* **BIT-REC**
 comprehension, *see also* **COMP**
 induction, *see also* **IND**, **PIND**
 maximization, *see also* **MAX**
 minimization, *see also* **MIN**
 replacement, *see also* **REPL**
 string induction, *see also* **SIND**
 string maximization, *see also* **SMAX**
 string minimization, *see also* **SMIN**
- B12'**, **B12''**: axioms of \overline{V}^0 , 125
 Barrington's Theorem, 333
BASIC, 71
 definition of, 257
 1-**BASIC**, 40
 2-**BASIC**, 95
 2-**BASIC**⁺, 129
 basic semantic definition, 20
 Bennett's Trick, 65, 68
bG₀, *see also* bounded depth **G**₀
 bibliography, 457–465
 bin(*X*), 85, 102, 293
 binary notation, 85
 binary search in **VPV**, 221
 bit definition, *see also* bit-defining axiom, 108
 bit graph, 101
 bit recursion axioms, *see also* **BIT-REC**
 bit-definable function, 108
 Σ_0^B bit definitions for **FAC**⁰, 108
 extension by, 109
BIT-REC, 222, 367
 Σ_0^B -**BIT-REC** in **TV**⁰, 222
 blank symbol \emptyset , 446
 BNR, 287, 339
 *p*BNR, 288
 *p*BNR characterizes **L**, 356
 2-BNR characterizes **AC**⁰(2), 308
 4-BNR, 3-BNR characterizes **FAC**⁰(6), 318
 5-BNR characterizes **NC**¹, 333
 Bondy's Theorem, 299
 Boolean sentence
 balanced, encoding of, 321
 value problem, *see also* BSVP
 Boolean Sentence Value Problem, *see also* BSVP
 bound variable, 19, 21
 Bounded Definability Theorem, 52
 bounded depth **Frege**, *see also* bounded depth **PK**
 bounded depth **G**, 195
 bounded depth **GTC**₀, 441
 bounded depth **PK**, 161
 definition of, 164
 bounded depth **PTK**, 430
 Bounded Depth Lower Bound Theorem, 165
 bounded formula, 42
 of \mathcal{L}_{S_2} , 256
 bounded induction scheme, 44
 bounded length induction, 265
 bounded number recursion, *see also* BNR
 bounded quantifier, 41
 Bounded Reverse Mathematics, 268
 open problems, 358
 bounded theory, 45
 bounding term, 45
bPK, *see also* bounded depth **PK**
 branching program, 334
 BSVP, *see also* MFV, 321, 334, 410, 414
- candidate solution, 229
 Cayley–Hamilton Theorem, 358
CC(**S**), 232
 cedent, 10
 extension cedent, 187
 characteristic function, 116
 circuit, 453
 encoding of, 335
 Monotone Circuit Value Problem, *see also* MCV
 Circuit Value Problem, 203, 335
 reduced to HornSat, 224
 class **C**
 definable in **VC**, 274
 definable in \overline{VC} , 280
 definable in \widehat{VC} , 275, 278
 represented in \mathcal{L}_{FC} , 279
 represented in $\mathcal{L}_{\widehat{VC}}$, 357
 closed formula, 19
 closed term, 19
 closure
 AC⁰-closure, 269
 Σ_0^B -closure, 109
co-C, 450
co-NL, 450
co-NP, 450

- Cobham's characterization, 137, 139, 140,
 201, 210
- collapse
 of *ACC* (conditional), 316
 of *PH* (conditional), 253
- COMP**
 Σ_0^p -comp, 130
 Δ_1^B -*COMP*, 222
 $\Sigma_0^B(\Phi)$ -*COMP*, 134
 definition of, 96
 multiple, *see also MULTICOMP*
 proves *IND*, *MIN*, 98
- Compactness Theorem
 predicate calculus, 34
 propositional, 16
 two-sorted logic, 89
- completeness, 73
- Completeness Lemma for *LK*, 25
- Completeness Theorem
*LK*², 89
PTK, 430
 Anchored *LK*, 30
 Anchored *PK*, 14
 Anchored, for *LK* with =, 33
 for *G*, 176
 for *LK*, derivational, 25
 for *LK*, Revised, 33
 for *PK*, 13
 for *PK*, derivational, 13
 major corollaries, 34
- complexity class, 65
- composition, 270
 of Σ_1 -definable functions, 51
- comprehension
 multiple, *see also MULTICOMP*
- comprehension axioms, *see also COMP*
- comprehension variable, 196
- computation, 447
 of ATM, 452
 of NTM, 449
- concatenation function $U *_t V$, 231
- Concatenation Recursion on Notation, 360
- concurrent random access machine
 (CRAM), 75
- configuration, 446
 functions encoding, 138
- connective, 9
 general, restricted, 177
 threshold Th_k , 428
- connectivity, *see also st-CONN*
R_{CONN}, 340
R_{UCONN}, 359
- for undirected graphs, 359
- consequent, 10
- conservative extension, 51
 $\mathcal{T}_1 \subset_{\text{cons}} \mathcal{T}_2$, 201
 assuming *COMP*, 110
 assuming *REPL*, 147
 w.r.t. Φ , 169
- Conservative Extension Lemma, 52
- Conservative Extension Theorem, 53
- conservativity
 Σ_{i+1}^B -conservativity of V^{i+1} over TV^i , 402
- constant symbol, 17
- contents, vii–x, xii
- contraction rule derived from cut, 12
- Count*₂, 306
- counting circuits, 326
- counting gate, 282
- counting quantifier, 282
- counting sequence, 283, 434
- Count*_{*m*}, 316
- CRN, 360
- curve, 310
- cut
 anchored cut, 14
 cut formula, 11
 cut rule, 11
 cut-free proof, 11
- daglike proof system, 162
- definability
 Σ_1^B vs. Σ_1^1 , 108
 Σ_0^B -definability, 110, 269
 bit-definable vs. definable, 109
 two-sorted, 107
- definability in V^∞ , 245
- Definability Theorem
 Σ_{i+1}^B -definable functions of TV^i , 248
 Σ_{i+1}^B -definable functions of V^{i+1} , 248
 $\Sigma_1^B(\mathcal{L}_{FP^{i+1}})$ -definable functions of VPV^i ,
 248
 Σ_0^B -definable functions of V^0 , 110
- Bounded Definability Theorem, 52
 for $\mathcal{I}\Delta_0$, 69
 for V^1 , 135, 156
 for VPV , 214
 for V^0 , 117
- definable function, *see also* bit-definable
 function, 50
 and definable predicate, 116
 bit-definable function, 108
from, 269
 two-sorted, 107

- definable predicate, 50
 - Δ_1^1 -, Δ_1^B -definable in \mathcal{T} , 115
- definable relation, 66
- definable search problem, 232
- defining axiom
 - bit-defining axiom, *see also* bit-defining axiom
- defining pair, 196
- defining triple, 412
- dependence degree, 196, 412
- depth of a formula, 164
- descriptive complexity theory, 73, 74
 - characterization of \mathcal{P} , 223
- determined variable and sequent, 177
- d - \mathbf{G}_0 , *see also* bounded depth \mathbf{G}_0
- direct connection language, 454
- Distance Problem, UDP, 359
- divisibility relation, 61
- division
 - $\lfloor X/Y \rfloor$ definable in $VTC^{0?}$, 360
 - number function $\lfloor x/y \rfloor$, 60
 - string function $X \div Y$, $\lfloor X/Y \rfloor$, 137
- DLogTime**, 449
- double-rail logic, 203
- DSpace**, 448
- DTime**, 447
- DTM, *see also* Turing machine

- E**, *see also* empty set, axiom
- eigenvariable, 22, 175
- Elimination Lemma
 - $\Sigma_1^B(\mathcal{L}_{FC})$, 280
 - $\Sigma_1^B(\mathcal{L}_{\overline{FC}})$, 278
 - FAC**⁰, 127
 - Row*, 114
- Elimination Theorem
 - First, 276
 - Second, 291
- empty set, 112
 - axiom for \emptyset , 91
- encoding
 - formalization vs propositional translation, 387
 - of balanced monotone Boolean sentences, 321
 - of circuits, 335
 - of graphs, 339
 - of proofs, 365
- endsequent, 10
- ePK**, *see also* extended **PK**, 382, 408
 - $TV^0 \vdash RFN_{ePK}$, 402
- equality axioms, 31
 - for **LK**, 32
 - for **LK**², 88
- Equality Theorem, 32
- equivalence, 10, 20
- EXP**, 447
- expansion
 - Σ_i^q -expansion of a formula, 399
- expansion of a model, 50
- extended **Frege**, *see also* extended **PK**
- extended **PK**, 161, 187
 - p-equivalence to \mathbf{G}_1^* , 187
- extended connection language, 320, 454
- Extension by Definition
 - Lemma for bit definition, 109
 - Theorem for two-sorted, 107
- Extension by Definition Theorem, 51
- extension cedent, 187
- extension variable, 187

- FAC**⁰
 - \mathcal{L}_{FAC^0} , 125
 - Σ_0^B -bit-definable, 108
 - characterization, 102
 - closed under Σ_0^B -definability, 112
 - closure under composition, definition by cases, 103
 - definable in V^0 , 110
- factoring, 242
- Fanin2*, 335
- finite axiomatizability
 - of TV^i , $\Sigma_0^B(V^i)$, 405
 - of TV^0 , 220
 - of **VACC** (conditional), 316
 - of V^i and TV^i , 243
 - of V^∞ (conditional), 245, 253
 - of **VP**, 202
 - of V^0 , 129
- finite sets as integers $bin(X)$, 85
- finite sets as strings, 81
- finitely satisfiable, 17
- first-order logic, 17
- first-order vocabulary, 17
- FO**, 74
- FO(COUNT)**, 282
- FO(M)**, 282
- formula, 9, 18
 - FLA_F** relation, 367
 - bounded, 42
 - bounded \mathcal{L}_{S_2} formulas, 256
 - bounded, two-sorted, 81
 - closed formula, 19
 - linearly bounded, 86

prenex form, 38
 provably computable, 373
 pseudo formula, 366
 QPC formula, 173
 QT, quantified threshold, 441
 quantifier-free, 35
 recognition in TC^0 , 366
 threshold, **PTK**, 429
 universal closure of, 24
 universal formula, 54
 formula classes
 $\Delta_0(\mathcal{L})$, $\Sigma_1(\mathcal{L})$, 51
 $\Sigma_i^B(\mathcal{L})$, $\Pi_i^B(\mathcal{L})$, 82
 $\Sigma_0^B(\Phi)$, 109
 Δ_0^B in a theory, 212
 Δ_0 , 42
 Σ_1^B -**Horn**, 224
 Σ_i^b , 70
 Σ_i^q , Π_i^q , 174
 Σ_1^1 -**Krom**, 343
 Σ_1 , 42
 Σ_1^1 , 82, 107
 $g\Sigma_i^B$, $g\Pi_i^B$, 142
single- Σ_1^B , 115
 Horn, 223
 Krom, 339, 343
 Formula Replacement Theorem, 21
FO(**THRESHOLD**), 282
 $FP^{\Sigma_i^p}$, 245
 $FP^{\Sigma_i^p}$ [*wit*, $O(1)$], 253
 $FP^{\Sigma_i^p}$ [*wit*, $O(g(n))$], 250
 free variable, 19, 21
 free variable normal form, 23, 90
 computable in polytime, 373
 for **G**, 175
 for proofs with **IND** rule, 154, 170
 free-cut-free proof, 14
 freely substitutable, 21
Frege systems, 161
 function
 arithmetical, 107
 bit graph, 101
 bounding term for function, 104
 definition by case, 103
 elimination from formulas, *see also* Transformation Lemma, Elimination Lemma
 graph, 101
 provably total function, 108
 two-sorted function, 101
 function class, 101, 270
FC and **C**, 116

and complexity class, **FC** and **C**, 270
 defining **FC** from **C**, 101
 function graph, 69
 function problem, 229
Fval, 323
G, 175
G^{*}, 175
KPG, 176
 cut-free **G**^{*}, 405
G_{*i*+1}^{*} *p*-simulates **G**_{*i*} w.r.t. $\Sigma_i^q \cup \Pi_i^q$, 179
G_{*i*} *p*-simulates **G**_{*i*+1}^{*}, 182
G_{*i*}^{*}: only cut prenex Σ_i^q formulas, 181
 $\hat{\mathbf{G}}_i^*$, 181
G_{*i*}, **G**_{*i*}^{*}, 179, 363
G_{*i*}, **G**_{*i*}^{*}: only cut Σ_i^q formulas, 180
 upper bound, 177
 Witnessing Problem, *see also* Witnessing Problem
G₀^{*}
 p-simulates **G**₀ w.r.t. prenex Σ_1^q , 180
 Replacement Lemma, 185
G₁^{*}
 p-equivalence to **ePK**, 187
 Witnessing Theorem, 186
GapL, 358
 general connective, 177
 Gentzen's **PK**, 10
 Gentzen's Midsequent Theorem, 180
 Gödel's Incompleteness Theorem, 43
 Grädel, 343
 Grädel's Theorem, 225
 graph
 st-CONN Problem, 339
 definition of, 101
 Distance Problem, UDP, 359
 encoding of, 339
 PATH problem, 351
 transitive closure, 343
 undirected, connectivity of, 359
 ground instance, 35
GTC, 441
 heap, 321
 Herbrand π disjunction, 180
 Herbrand Theorem, 35, 36, 267
 proving Witnessing Theorem for V^0 , 127
 Second Form, 54
 two-sorted logic, 90
 Horn formula, 223
 HornSat algorithm, 224
ID₀, 46

- $I\Delta_0$, 42, 49
 $\overline{I\Delta_0}$, 54, 56
 alternative axioms, 44
 and V^0 , 99
 Definability Theorem, 69
 defining $BIT(i, x)$, 64
 defining $BIT(i, x)$, $y = 2^x$, 59
 defining $y = 2^x$, 59, 62
 provably total function, 51
 Immerman–Szelepcsényi Theorem, 343, 450
IND, *see also* **PIND**, **SIND**, **LIND**, 41
 Σ_{i+1}^B -**IND** \vdash Π_i^B -**REPL**, 144
 Δ_i^B -**IND**, 222
 Φ -**IND** rule, 150
 $\Sigma_0^B(\Sigma_i^B \cup \Pi_i^B)$ -**IND** in V^i , 134
 bounded induction, 44
 definition of, 97
 implied by **COMP**, 98
 induction in V^i , 133
 strong induction, 44
 Independence of PHP from V^0 , 168
 index, 465–479
 induction
 X -**IND**, in V^0 , 98
 induction axioms, *see also* **IND**, **PIND**,
SIND, **LIND**
 bounded length induction, 265
 inference rule for **LK**, 22
 initial sequents, 10
 interpreting **IOPEN** in V^1 , 137
 introducing new symbols, 50
 inversion principle, 13
IOPEN, 42
 interpreted in V^1 , 137
 $I\Sigma_0^{1,b}$, 130
 $I\Sigma_1$, 42
 provably total function, 51
ITERATION, 230
 combining, 234
 Jeřábek, 157
 Jordan Curve Theorem, 268, 303, 309
 König's Lemma, 16
KPG, 176
 KPT Witnessing Theorem, 238, 253
 Krom formula, 339, 343
 Σ_1^1 -**Krom** formula, 343
 Representation Theorem, 345
 Krom-SAT, 339, 343
L, 339, 351, 447
 characterized by BNR, 287, 356
 definition of, 448
 theories for, *see also* **VL**, **VLV**
#L, 358
 language, 17
 languages of theories, *see also* vocabularies
 layered circuit, 335
 least number principle, *see also* **MIN**
left, *see also* projection function
 length function for number, $|x|$, 64
 length function for string, $|X|$, 76
 length induction axioms, *see also* **LIND**
 length of a **G** proof (upper bound), 177
 length of a sequent, 177
LH, 75, 83, 455
 limited recursion, 139, 210, 268, 287
 limited subtraction, 60, 220
LIND, 257
 two-sorted, 338
 Lind's Characterization of **FL**, 356
 linear formulas, 87
 linear time hierarchy, *see also* **LTH**
 linearly bounded formula, 86
LinTime, 66
LK, 21
LK- Φ proof, 24
 anchored proof, 30, 33
 derivational soundness, completeness, 25
 equality axioms, 32
 revised soundness, completeness, 33
 Soundness Theorem, 22
LK with =
 Anchored Completeness Theorem, 33
 revised definition, 33
LK², 87, 88
LK²+**IND**, 150
LK²-**TV**¹, 233, 378
LK²-**VNC**¹, 411
LK²- V^0 , 411
LK²- \tilde{V}^1 , 150
LK²- \tilde{V}^0 , 170
 soundness and completeness, 89
 local search, 229
 log time hierarchy, *see also* **LH**
 logical axiom, 11
 logical consequence, 10, 21, 80
 $\Sigma_j^B(V^i)$, 403
 of a set of sequents, 13
 logspace, *see also* **L**
 Löwenheim, 34
LTH, 65, 66, 86
FLTH, 69
 defining number function $x \cdot y$, 87

- definition of, 452
- LTH** Theorem, 67
- majority gate, 282
- majority gates, 455
- majority quantifier, 282
- many-one reducible, 229
- mappings \flat and \sharp , 258
- MAX**, *see also* **SMAX**
 - Σ_i^B -**MAX** in V^i , 133
 - Σ_1^B -**MAX** in \tilde{V}^1 , 150
 - $\Sigma_0^B(\Sigma_i^B \cup \Pi_i^B)$ -**MAX** in V^i , 134
 - definition of, 97
- maximization axioms, *see also* **MAX**, **SMAX**
- MCV, 202, 203, 272
- MFV, 321
 - MFV variable, 412
 - MFV^i , $\delta_{MFV}^i(a, G, I, Y)$, 411
 - propositional translation, 382
- Midsequent Theorem, 180
- MIN**, *see also* **SMIN**
 - X -**MIN** provable in V^0 , 97
 - Σ_i^B -**MIN** in V^i , 133
 - $\Sigma_0^B(\Sigma_i^B \cup \Pi_i^B)$ -**MIN** in V^i , 134
 - definition, 44
 - definition of, 97
 - implied by **COMP**, 98
- minimal theory, 268
 - VP**, 210
 - for polytime, 201
- minimization axioms, *see also* **MIN**, **SMIN**
- mod_m
 - Mod_m , 314
 - δ_{MOD_m} , 314
 - mod'_m , 314
- model
 - expansion of a model, 50
 - term model, 29
- model, $\mathcal{M} \models A$, 19
- MODULO**, 303, 456
- Modulo m Counting Principle, 316
 - $V^0(m) \vdash \forall a \forall X \text{Count}_p(a, X)?$, 360
- modulo m gates, 455
- Modulo 2 Counting Principle, 306
- modulo gate, 303
- Monotone Circuit Value Problem, *see also* MCV
- Monotone Formula Value Problem, *see also* MFV
- Multi Tape Theorem, 448
- MULTICOMP**
 - definition of, 114
 - revisited, 134
- Multiple Comprehension Lemma, 114
- multiplication, 73
 - $R_\times(X, Y, Z)$, 85
 - $R_\times(X, Y, Z)$ not in AC^0 , 102
 - $X \times Y$, 361
 - $X \times Y$ in V^1 , 136
 - $X \times Y$ in VTC^0 , 293
 - $X \times Y$ not Σ_1^B -definable in V^0 , 109
- multivalued function, 229
- NC**¹, 319, 456
 - algorithm for BSVP, 415
 - characterized by 5-BNR, 333
 - theories for, *see also* **VNC**¹, **VNC**¹**V**
 - Witnessing Problem for G_0^* , G_0 , 428
- NC** hierarchy, 319, 335
- NC** ^{k} , 319, 320
 - definition of, 455
- Nepomnjašćij's Theorem, 68
- NL**, 339
 - $NL \subseteq LTH$, 68
 - closed under complement, 339
 - definition of, 450
 - theories for, *see also* **VNL**, **V**¹-**KROM**
- NLinTime**, 66, 451
- DLogTime**, 449
- NLogTime**, 83
- Nondeterministic logspace, *see also* **NL**
- nondeterministic Turing machine, *see also* NTM
- nonlogical axiom, 13
- NOTM, 451
- NP**, 456
 - definition of, 450
- NP**^C, 451
- NP** ^{Σ_i^B} , 65
- NSpace**, 450
- NTime**, 450
- NTimeSpace**, 68
- NTM
 - definition of, 449
- number quantifier, 77
- number recursion, *see also* BNR
- number summation, *see also* summation
- number term, 77
- number variable, 76, 87
- numeral, \underline{n} , 40
- NUMONES, 326
 - propositional translation, 434
 - relation, in $I\Delta_0$, 65
- numones*, 282

- Numones'*, 285
numones', 286
Numones, 284
 δ_{NUM} , 283
 defining TC^0 , 283
 for showing $V^1 = \tilde{V}^1$, 149
 in VNC^1 , 326
 quantifier-free definition, 286
- object assignment, 19, 79
OPEN, 42
 oracle Turing machine, *see also* OTM
 ordering for string, *see also* string \leq
 OTM, 451
- P**, **FP**, *see also* polytime
 p-bounded function, 45, 101
 in \mathcal{T} , 105
 p-bounded theory, 45
 two-sorted, 105
 p-equivalence, 161
 G_{i+1}^* and G_i w.r.t. Σ_i^q , 408
 G_{i+1}^* and G_i w.r.t. Σ_i^q , 182
PK and **PTK**, 431
ePK and G_1^* , 187
 p-simulation
 follow from RFN, 407
 pairing function, 113
 $\langle x, Y \rangle$ and $\langle X, Y \rangle$, 243
 palindrome, 75, 83
 parameter variable, 23
 Parikh's Theorem, 44, 45, 116, 141
 alternative proof for IA_0 , 58
 two-sorted, 104, 105
 parity, 303
PARITY(X), 240
Parity(x, Y), 304
parity(X), 118
parity(X) definable in V^1 , 136
 φ_{parity} , 118, 270, 304
PARITY, 118, 456
PARITY not in $AC^0/poly$, 75
 Σ_1^B -Horn-formula *Parity*_{Horn}(X), 225
 not in AC^0 , 118
 separates V^0 and VTC^0 , 287
 Parity Principle, 306
 partial order, 298
 PATH problem, 339, 351
P^C, 451
pd, *see also* predecessor function
 Peano Arithmetic, 39
 Peano Arithmetic **PA**, 41
 pebbling game, 415
- PH**, 65, 174
 V^∞ vs **PH**, 253
 definition of, 451
 PHP, 163, 164, 268, 440
 $V^0(m) \vdash PPHP?$, 360
 in VTC^0 , **PTK**, 291
 in two-sorted logic, 167
 independence from V^0 , 168
 onto PHP, **OPHP**, 306, 317
 separates V^0 and VTC^0 , 287
 Pigeonhole Principle, *see also* PHP
PIND, 71, 257
PK, 10, 160
 derivational soundness completeness, 13
 p-simulates **PTK**, 431
 proof, 10, 12
 from assumptions, 13
 Replacement Lemma, 163
 short **PK***-proofs of true sentences, 370
 soundness and completeness, 12
 treelike *p*-simulates daglike, 162
 PK Simulation Theorem, 424
PLS, 186, 228, 229
 $PLS^{\Sigma_{i-1}^P}$ and TV^i , 249
P^{NP}, 186
 polynomial equivalence, *see also* p-equivalence
 Polynomial Local Search, *see also* **PLS**
 polynomial simulation, *see also* p-simulation
 polynomial time, *see also* polytime
 polynomial time hierarchy, *see also* **PH**
 polynomial-bounded theory, *see also* p-bounded theory
 polynomially bounded function, *see also* p-bounded function
 polynomially bounded proof system, 159, 160
 polynomially induction axioms, *see also* **PIND**
 polytime, 65, 133, 447
P/poly, 254
 characterized by Σ_1^B -Horn, 225
 characterized by V^1 , 135
 Cobham's characterization, 139
 definition of, 447
 descriptive characterization of, 223
 function, relation, 135
 theories for, *see also* TV^0 , V^1 , **VP**, *see also* V^1 -HORN, **VPV**, **VC**
 theories for subclass, 267, 272, 280
 predecessor function
 axiom in IA_0 , 44

- for number, *pd*, 125
- predicate calculus, 17
 - semantics, 19
 - syntax, 17
- preface, xiii–xv
- prenex form, 38
- Prenex Form Theorem, 38
- prenex formula
 - prenexification in TC^0 , 386
- prime factorization, 148
- prime recognition, 148
- principal formula, 11, 22
- principal formulas, 175
- principle
 - least number, *see also MIN*
- probabilistic polytime, 242
- projection function, 113
- proof
 - LK^2 , 88
 - PK , 10, 12
 - $PRF_{\mathcal{F}}$ relation, 367
 - anchored proof, *see also* anchored proof, 14
 - encodings of, 365
 - extended PK , 187
 - from assumptions, 13
 - provably computable, 373
 - verification in TC^0 , 364
- proof system, 159, 160
 - existence of polynomially bounded proof system $\Leftrightarrow NP = co-NP$, 161
- QPC, 175
 - treelike vs daglike, 162
- propositional calculus, 9
- Propositional Compactness Theorem, 16
- propositional formula, 9
- propositional proof system, *see also* proof system
- propositional translation
 - V^0 to bPK , 165
 - computable in TC^0 , 373
 - for V^i , 191
 - for bounded \mathcal{L}_A^2 formulas, 191
 - for MFV, 382
 - formalization of, 364
 - of Σ_0^B (*numones'*) formulas, 434
 - of *NUMONES*, 434
 - translating Σ_0^B formulas, 166
 - vs formalization, 387
- propositional variable, 9
- propositionally unsatisfiable, 35
- prototype, 180
- provably collapse, 253
- provably computable (formulas or proofs), 373
- provably total function, 51
 - closed under composition, 108
 - two-sorted, 108
- pseudo formula, 366
- PSPACE**, 174, 447, 448
 - definition of, 448
- PSPACE/poly**, 174
- PTK**, 291, 428
 - PK p-simulates PTK , 431
 - bounded depth, $bPTK$, d - PTK , 430
 - soundness, completeness, 430
- QPC, 173
- QPC proof system, 175
- QPC sentence, 174
- quantified propositional calculus, *see also* QPC
- quantified threshold formulas, 441
- quantifier, 18
 - $\exists X \leq T, \forall X \leq T$, 219
 - bounded, 41
 - in two-sorted logic, 77
 - sharply bounded, 70
 - sharply bounded quantifiers, 256
- quantifier-free formula, 35
- Razborov, 304
- REACH**, 341
- reachability, φ -*Reach*, 343
- Σ_0^B -**Rec**, 353
- recursion, 268
 - limited, *see also* limited recursion
- recursively enumerable, 35
- reduction, 73
 - $\leq AC^0$ for search problems, 229
 - $\leq TC^0$ for search problems, 409
 - AC^0 -reduction, 267, 269
- AC^0 -reduction, 270
- Reflection Principle, *see also* RFN, 159
- relation
 - representable or definable, 66, 83
- remainder function, 60
- remainder string function, 137
- REPL**, 142, 204
 - Σ_{i+1}^B -**IND** \vdash Π_i^B -**REPL**, 144
 - Π_i^B -**REPL** \vdash Σ_{i+1}^B -**REPL**, 144
 - $V^0 \not\vdash \Sigma_0^B$ -**REPL**, 240
 - Σ_0^B -**REPL** and VPV , 242
 - $g^{\Sigma_i^B}$ -**REPL** in V^i , 145

- and conservative extensions, 147
- definition of, 143
- replacement axioms, *see also REPL*
- Replacement Lemma
 - for G_0^* , 185
 - for PK , 163
- represent sets as binary strings, 81
- representable relation, 66, 83
- Representation Theorem
 - Σ_1^B , 86
 - Σ_1^B , 86
 - Σ_0^B , 84
 - for Σ_1^B -Horn formulas, 225
- Representation Theorems, 81
- restricted connective, 177
- RFN, 363, 382
 - Σ_{i+1}^q -RFN_{cut-free} G^* , 405
 - $TV^i \vdash \Sigma_{i+1}^q$ -RFN G_{i+1}^* , Π_{i+1}^q -RFN G_i , 402
 - $V^i \vdash \Pi_i^q$ -RFN G_{i-1} , Σ_i^q -RFN G_i^* , 400
- axiomatize $\Sigma_j^B(V^i)$, 403
- axiomatize TV^i , 404
- definition of, 396
- for d -PTK, 433
- for ePK , 402
- for PK , 421
- for subsystems of G , 396
- prove p-simulations, 407
- treelike vs daglike, 398
- right*, *see also* projection function
- ring, 43
- Row, 114
- RSUV isomorphism, 256, 258
 - between S_2^i and V^i , 263
 - definition of, 259
- rudimentary function, 131
- rule
 - of PK , 11
- rule of inference, 11
- Ruzzo, 320, 336
- S_2^1 , 156, 256
 - $S_2^1(BIT)$, 258
 - provably total function, 51
- S_2^i , 256, 258
- S_2^i hierarchy, 70
- SAT
 - Krom-SAT, *see also* Krom-SAT
- satisfaction, 10
 - $\mathcal{M} \models A$, 19
 - $\mathcal{M} \models A[\sigma]$, 20
 - $\mathcal{M} \models \Phi[\sigma]$, $\mathcal{M} \models \Phi$, 20
 - for a sequent, 10
- satisfiability problems
 - HornSat, 224
- satisfiability relation
 - $(Z \models_0 X)$, 410
 - $(Z \models X)$, 382
 - Δ_1^B -definable in TV^0 , 384
 - Δ_1^B -definable in VNC^1 , 421
 - for Σ_i^q , Π_i^q formulas, 385
- satisfiable set of formulas, 10
- Savitch's Theorem, 450
- search problem, 229
 - $CC(S)$, 232
 - Σ_i^B -definable in TV^i , 249
 - Σ_{i+1}^B -definable in V^i , 250
 - Σ_{i+1}^B -definable in VPI^i , 253
 - Σ_j^B -definable in V^i, TV^i , 250
 - definable in a theory, 232
- second-order logic, 73
- semantics of first-order sequents, 22
- semantics of predicate calculus, 19
- semantics of two-sorted logic, 78
- sentence, 19
 - \forall sentence, 35
- SEQ, 283
- $seq(x, Z)$, *see also* sequence, coding
- sequence
 - encode a sequence of numbers, 115
 - encode a sequence of string, 115
- sequent, 10
 - active sequent, 27
 - determined sequent, 177
 - endsequent, 10
 - initial sequents, 10
 - logical consequence of, 13
 - semantics of, 10
 - valid sequent, 10
- sequent length, 177
- set
 - empty, *see also* empty set
- set variable, 76
- sharply bounded quantifier, 70
- sharply bounded quantifiers, 256
- SIND**
 - Δ_i^B -SIND, 222
 - Φ -SIND rule, 233
 - SIND' rule, 378
 - definition of, 218
- single string quantifier, 115
- singleton set, *see also* POW2
- Skolem, 34
- Skolem functions, 35, 55
- smash function #, 70

SMAX

definition of, 219
 prove **BIT-REC**, 222

SMIN

definition of, 219

Smolensky, 304

sorting, 287

soundness principle, *see also* RFN

Soundness Theorem

LK², 89

PTK, 430

for **G**, 176

for **LK**, 22

for **LK**, derivational, 25

for **LK**, Revised, 33

for **LK**²+**IND**, 151

for **PK**, 13

for **PK**, derivational, 13

space constructible function, 448

Space Hierarchy Theorem, 448

Speed-up Theorem, 447

square root function, 60

standard model \mathbb{N}_2 , 79

standard model, \mathbb{N} , 20, 40

st-CONN, 339

strict Σ_1^b , 156

string comprehension, 270

string function

addition, *see also* addition

definability, *see also* definability

division, *see also* division

empty set, *see also* empty set

encoding Turing machine configurations,
see also configuration, functions encoding

multiplication, *see also* multiplication

successor, *see also* successor function

string induction axioms, *see also* **SIND**

string maximization axioms, *see also* **SMAX**

string minimization axioms, *see also* **SMIN**

string ordering $X \leq Y$, 219

string quantifier, 77

$\exists X \leq T, \forall Y \leq T$, 219

string term, 77

string variable, 76, 87

strong induction, 44

structure, 19

weak structure, 31

\mathcal{L}_A^2 -structure, 78

student-teacher, 238

subformula property, 14, 15

of **LK**²+**IND**, 153

of anchored **LK**, 33

of anchored **LK**², 89

provable in **VTC**⁰, 373

sublinear time classes, 449

substitution, 21

Substitution Lemma, 185

Substitution Theorem, 21

substring function

$X[i, j]$, 366

subtraction $Z \dot{-} Y$, 220

succedent, 10

successor function $S(X)$, 112, 218

successor relation $S(X, Y)$, 378

Sum, 294

summation, 288, 289

syntax of predicate calculus, 17

Szpilrajn's Theorem, 298

T_2^i , 71, 256, 258

TA, 40

tagged formula, 183

Tape Compression Theorem, 448

target formula, 175

TAUT, 160

tautology, 10

TC⁰, 70, 282

TC⁰/*poly*, 282

closed under summation, 288

computing propositional translation, 373

contains $X \times Y$, 293

contains **ACC**, 304

definition of, 456

finite iterations, 269

functions for verifying proofs, 364

sorting in **TC**⁰, 287

theories for, *see also* **VTC**⁰, **VTC**⁰**V**

verifying proofs in **TC**⁰, 364

term, 18

bounding term for function, 104

closed term, 19

in two-sorted logic, 77

term model, 29

Theorem

Grädel, 225

theorem, 40

theory, 39, 90

p-bounded theory, 45

universal theory, 54

threshold gate, 282

threshold gates, 456

threshold logic, 428

time constructible function, 448

Time Hierarchy Theorem, 448

- total order, 298
 Transformation Lemma
 Σ_1^B , using **REPL**, 145
 Σ_0^B , using (bit) definitions, 111
 transitive closure, 343
ContainTC, 343
 translation
 \flat translation, 262
 \sharp translation, 260
 propositional, *see also* propositional translation
 Translation Theorem, 363
 for TV^i , 377
 for TV^0 , 382
 for V^i , 192
 for V^i , formalization in \overline{VTC}^0 , 377
 for VNC^1 , 411
 for VTC^0 , 442
 for V^0 , 168
 all bounded theorems, 195
 proof of, 170
 for V^0 , formalization in \overline{VTC}^0 , 375
 for $V^0(\text{numones}')$, 442
 tree recursion, *see also* TreeRec
 treelike proof system, 162
 Σ_0^B -TreeRec, 325
 True Arithmetic, 40
 truth assignment, 9
 \mathcal{L} -truth assignment, 35
 truth definitions, 383
 truth value, 9
 tupling function, 113
 Turing machine
 alternating TM, *see also* ATM
 definition of, 445
 nondeterministic oracle TM, *see also* NOTM
 NOTM
 nondeterministic TM, *see also* NTM
 oracle TM, *see also* OTM
 TV^0 , 217, 367
 $TV^0 = VP$, 220
 $\vdash \Sigma_0^B$ -BIT-REC, 222
 $\vdash RFN_{ePK}$, 402
 $\vdash \Delta_1^B$ -SIND, 223
 V^1 is Σ_1^B -conservative over TV^0 , 220
 finite axiomatizable, 220
 TV^1 , 228
 $TV^1(VPV)$, 220
 Witnessing Theorem, 233
 TV^i , 217, 218, 363
 $TV^i(VPV)$, 220
 $TV^i(VPV)$ conservative over TV^i , 220
 $V^i \subseteq TV^i$, 219
 $V^i \subseteq TV^i$, 244
 $\vdash \Sigma_i^B$ -IND, 218
 $\vdash \Sigma_{i+1}^q$ -RFN $_{G_{i+1}^*}$, Π_{i+1}^q -RFN $_{G_i}$, 402
 $\vdash \Sigma_i^B$ -SMIN, SMAX, 219
 Σ_i^B -definable search problems of, 249
 Σ_j^B -definable search problems of, 250
 $V^{i+1} \Sigma_{i+1}^B$ -conservative over TV^i , 249
 axiomatized by RFN, 404
 finite axiomatizable, 405
 finitely axiomatizable, 243
 Translation Theorem, 377
 two-sorted classes, 73, 80
LTH, 86
 two-sorted function, 101
 two-sorted logic, 73, 76
 interpreted as single-sorted, 91
 semantics, 78
 syntax, 76
 two-sorted theory, 90
 definability, 107
 Tychonoff's Theorem, 17
 U^1 , 338
 uniformity, 453
 universal closure, 24
 universal conservative extension, 124
 universal formula, 54
 universal function, 243
 universal theory, 54
 Witnessing Theorem, 211
 universe, 19
 unsatisfiable set of formulas, 10
 upper bound for G proofs, 177
 V^0 , 73, 95, 267
 $V^0(\text{Row})$, 114
 $V^0(\text{True}\Sigma_0^B)$, 128
 $V^0(\emptyset, S, +)$, 112
 $V^0 \not\vdash \text{Count}_2$, 307
 $V^0 \not\vdash \text{OPHP}$, 306
 $V^0 \not\vdash \Sigma_0^B$ -REPL, 240
 Σ_0^B -definable functions, 110
 $V^0(2)$ not Σ_0^B -conservative over V^0 , 308, 318
 \overline{V}^0 , 124
 \tilde{V}^0 , 169
 conservative over IA_0 , 99
 definability in V^0 , 106
 Definability Theorem, 117
 finite axiomatizability, 129
 independence of PHP, 168

- properly contained in V^1 , 136
 properly in $V^0(m)$, 318
 properly in VTC^0 , 287
 properly in $V^0(2)$, 308
 proves X -IND, 98
 proves X -MIN, 97
 Translation Theorem, 168
 all bounded theorems, 195
 formalization in \overline{VTC}^0 , 375
 proof of, 170
 Witnessing Theorem, 117
 $V^0(2)$, 303, 304
 $\overline{V^0(2)}$, 305
 $\widehat{V^0(2)}$, 305
 Definability Theorem, 305
 not Σ_0^B -conservative over V^0 , 308, 318
 properly contains V^0 , 308
 proves $Count_2$, 307
 proves Jordan Curve Theorem, 311
 proves OPHP, 306
 $V^0(m)$, 314
 $V^0(m) \vdash PHP?$, 360
 $V^0(m) \vdash \forall a \forall X Count_p(a, X)?$, 360
 $\vdash Count_m$, 317
 $\vdash Count_{m'}$ for $\gcd(m, m') > 1$, 317
 $\overline{V^0(m)}$, 315
 $\widehat{V^0(m)}$, 314
 Definability Theorem, 315
 properly contains V^0 , 318
 $V^0(numones')$, 434
 Translation Theorem, 442
 V^1 , 133
 $\vdash \Sigma_1^B$ -REPL, 144
 $\vdash \Delta_1^B$ -SIND, 223
 Σ_1^B -conservative over TV^0 , 220
 Σ_1^B -conservative over VP , 217
 $V^1(VPV)$, 213
 \tilde{V}^1 , 149
 and VPV , 213
 characterizes P , 135
 Definability Theorem, 135
 extended by polytime functions, 145
 interprets $IOPEN$, 137
 prime factorization, 148
 prime recognition, 148
 properly contains V^0 , 136
 Witnessing Theorem, 147
 proof of, 154
 V_1^1 , 156
 V^1 -KROM, 339, 343
 = VNL , 347
 definition, 346
 V^i , 96, 363
 $V^i \subseteq TV^i$, 219
 $V^i \subseteq TV^i$, 244
 $\vdash \Sigma_0^B(\Sigma_i^B \cup \Pi_i^B)$ -COMP, 134
 $\vdash \Sigma_i^B$ -IND, MIN, MAX, 99, 133
 $\vdash g\Sigma_i^B$ -REPL, 145
 $\vdash \Pi_i^q$ -RFN $_{G_{i-1}}$, Σ_i^q -RFN $_{G_i^*}$, 400
 Σ_{i+1}^B -definable search problems of, 250
 Σ_j^B -definable search problems of, 250
 $V^i(VPV)$, 213
 $V^{i+1} \Sigma_{i+1}^B$ -conservative over TV^i , 249
 finitely axiomatizable, 243
 Translation Theorem, 192
 formalization in \overline{VTC}^0 , 377
 V^∞ , 245
 vs PH , 253
 $VAC^0(2)V$, 303, 308
 $VAC^0(6)V$, 303, 318
 $VACC$, 315
 Definability Theorem, 315
 VAC^k and VNC^k , 337
 valid, 10
 valid formula, 10, 20
 valid QPC formula, 174
 valid sequent, 10, 22
 variable, 18
 bound variable, 21
 extension variable, 187
 free and bound in two-sorted, 87
 free and bound variable, 19, 21
 in two-sorted logic, 76
 parameter variable, 23
 propositional, 9
 VVC , 267
 $\vdash COMP, IND, MIN$, 274
 \overline{VC} , 278
 \widehat{VC} , 274
 definition, 273
 proves $COMP, IND, MIN$, 279
 V^1 -HORN, 223
 = VP , 226
 definition of, 226
 VL , 339, 351
 = Σ_0^B -Rec, 354
 $\stackrel{?}{=} VSL$, 358
 contains VNC^1 , 354
 Definability Theorem, 353
 VLV , 339, 356
 VNC^1 , 323
 = Σ_0^B -TreeRec, 325

- Δ_1^B -defines ($Z \models X$), 421
 contains VTC^0 , 326
 Definability Theorem, 324
 in VL , 354
 proves RFN_{PK} , 424
 Translation Theorem, 411
 VNC^1V , 333
 VNL , 339, 341
 $= V^1\text{-}KROM$, 347
 \overline{VNL} , 342
 Definability Theorem, 342
 vocabularies
 \mathcal{L}_A , 18, 40
 \mathcal{L}_{FAC^0} , 125
 \mathcal{L}_{FO} , 74
 \mathcal{L}_{Δ_0} , 56
 \mathcal{L}_{FP} , 210
 \mathcal{L}_{FP^i} , 246
 \mathcal{L}_{S_2} , 70, 256
 $\mathcal{L}_{S_2}^+$, 262
 \mathcal{L}_A^2 , 76
 \mathcal{L}^+ , 260
 vocabulary, 17
 VP , 201, 267, 273
 $= V^1\text{-}HORN$, 226
 $TV^0 = VP$, 220
 $VP \subseteq V^0 + \Sigma_0^B\text{-}BIT\text{-}REC$, 223
 V^1 is Σ_1^B -conservative over VP , 217
 VPV conservative over VP , 214
 \widehat{VP} , 207
 finite axiomatizable, 202
 minimal theory for P , 210
 VPV , 210
 $\vdash \Sigma_0^B(\mathcal{L}_{FP})\text{-}COMP, IND, MIN, MAX$,
 212
 $\vdash \Delta_1^B\text{-}SIND$, 222
 $\vdash \Sigma_0^B(\mathcal{L}_{FP})\text{-}SIND$, 221
 $\Sigma_0^B\text{-}REPL$ and VPV , 242
 and $V^1(VPV)$, 214
 and V^1 , 213
 binary search, 221
 conservative over VP , 214
 Definability Theorem, 214
 definition of, 211
 Witnessing Theorem, 212
 VPV^i , 245
 $\vdash \Sigma_0^B(\mathcal{L}_{FP^i})\text{-}COMP, IND, MIN, MAX$,
 247
 Σ_{i+1}^B -definable search problems of, 253
 definition of, 246
 VSL , 358
 VTC^0 , 281, 403
 Δ_1^B -defines $PRF_{\mathcal{F}}$, 368
 VTC^0V , 287
 $\overline{VTC^0}$, 286
 defines propositional translations, 375
 $V^0(\text{numones}')$, 434
 can define $\lfloor X/Y \rfloor?$, 360
 Definability Theorem, 286
 defining $X \times Y$, 293
 definition, 283
 in VNC^1 , 326
 properly extends V^0 , 287
 proves PHP, 291
 sorting in VTC^0 , 287
 Translation Theorem, 442
 VTC^0V , 289
 weak structure, 31
 witness query, 250
 witnessing functions, 117
 Witnessing Problem
 definition of, 409
 for G , 408
 for G_1^* , 363
 for G_0^*, G_0 , 428
 Witnessing Theorem
 for G_1^* , 186
 for TV^1 , 233
 for V^1 , 147
 proof of, 154
 for VPV , 212
 for V^0 , 95, 117, 118
 alternative proof of, 127
 proof of, 119
 for universal theories, 211
 KPT, *see also* KPT Witnessing Theorem
 Zambella, 130
 Zorn's Lemma, 17