In the last decade methods and techniques based on supersymmetry have provided deep insights in quantum chromodynamics and other non-supersymmetric gauge theories at strong coupling. This book summarizes major advances in critical solitons in supersymmetric theories, and their implications for understanding basic dynamical regularities of non-supersymmetric theories.

After an extended introduction on the theory of critical solitons, including a historical introduction, the authors focus on three topics: non-Abelian strings and confined monopoles; reducing the level of supersymmetry; and domain walls as D brane prototypes. They also provide a thorough review of issues at the cutting edge, such as non-Abelian flux tubes. The book presents an extensive summary of the current literature so that researchers in this field can understand the background and related issues.

Mikhail Shifman is the Ida Cohen Fine Professor of Physics at the University of Minnesota, and is one of the world leading experts on quantum chromodynamics and non-perturbative supersymmetry. In 1999 he received the Sakurai Prize for Theoretical Particle Physics, and in 2006 he was awarded the Julius Edgar Lilienfeld Prize for outstanding contributions to physics. He is the author of several books, over 300 scientific publications, and a number of popular articles and articles on the history of high-energy physics.

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Supersymmetric Solitons

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### Abbreviations

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<td>AdS</td>
<td>Anti de Sitter</td>
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<tr>
<td>ANO</td>
<td>Abrikosov–Nielsen–Olesen</td>
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<td>BPS</td>
<td>Bogomol'nyi–Prasad–Sommerfield</td>
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<td>CC</td>
<td>Central Charge</td>
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<td>CMS</td>
<td>Curve(s) of the Marginal Stability</td>
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<td>CFT</td>
<td>Conformal Field Theory</td>
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<td>CFIV</td>
<td>Cecotti–Fendley–Intriligator–Vafa</td>
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<tr>
<td>FI</td>
<td>Fayet–Iliopoulos</td>
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<tr>
<td>IR</td>
<td>Infrared</td>
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<tr>
<td>NSVZ</td>
<td>Navikov–Shifman–Vainshtein–Zakharov</td>
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<td>QCD</td>
<td>Quantum Chromodynamics</td>
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<td>SUSY</td>
<td>Supersymmetry, Supersymmetric</td>
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<td>SQCD</td>
<td>Supersymmetric Quantum Chromodynamics</td>
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<td>SQED</td>
<td>Supersymmetric Quantum Electrodynamics</td>
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<tr>
<td>UV</td>
<td>Ultraviolet</td>
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<tr>
<td>VEV</td>
<td>Vacuum Expectation Value</td>
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