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Introduction

It is well known that supersymmetric theories may have Bogomol'nyi–Prasad–Sommerfield (BPS) sectors in which some data can be computed at strong coupling even when the full theory is not solvable. Historically, this is how the first exact results on particle spectra were obtained [1]. Seiberg–Witten's breakthrough results [2, 3] in the mid 1990s gave an additional motivation to the studies of the BPS sectors.

BPS solitons can emerge in those supersymmetric theories in which superalgebras are centrally extended. In many instances the corresponding central charges are seen at the classical level. In some interesting models central charges appear as quantum anomalies.

First studies of BPS solitons (sometimes referred to as critical solitons) in supersymmetric theories at weak coupling date back to the 1970s. De Vega and Schaposnik were the first to point out [4] that a model in which classical equations of motion can be reduced to first-order Bogomol'nyi–Prasad–Sommerfeld (BPS) equations [5, 6] is, in fact, a bosonic reduction of a supersymmetric theory. Already in 1977 critical soliton solutions were obtained in the superfield form in some two-dimensional models [7]. In the same year miraculous cancellations occurring in calculations of quantum corrections to soliton masses were noted in [8] (see also [9]). It was observed that for BPS solitons the boson and fermion modes are degenerate and their number is balanced. It was believed (incorrectly, we hasten to add) that the soliton masses receive no quantum corrections. The modern – correct – version of this statement is as follows: if a soliton is BPS-saturated at the classical level and belongs to a shortened supermultiplet, it stays BPS-saturated after quantum corrections, and its mass exactly coincides with the central charge it saturates. The latter may or may not be renormalized. Often – but not always – central charges that do not vanish at the classical level and have quantum anomalies *are* renormalized. Those that emerge as anomalies and have no classical part typically receive no renormalizations. In many instances holomorphy protects central charges against renormalizations.

Critical solitons play a special role in gauge field theories. Numerous parallels between such solitonic objects and basic elements of string theory have been revealed in recent years. At first, the relation between string theory and supersymmetric gauge theories was mostly a "one-way street" – from strings to field theory. Now it is becoming exceedingly more evident that field-theoretic methods and results, in their turn, provide insights in string theory.

String theory, which emerged from dual hadronic models in the late 1960s and 70s, elevated to the "theory of everything" in the 1980s and 90s when it experienced an unprecedented expansion, has seemingly entered a "return-to-roots" stage. The task of finding solutions to "down-to-earth" problems of QCD and other gauge theories by using results and techniques of string/*D*-brane theory is currently recognized by many as one of the most important and exciting goals of the community. In this area the internal logic of development of string theory is fertilized by insights and hints obtained from field theory. In fact, this is a very healthy process of cross-fertilization.

If supersymmetric gauge theories are, in a sense, dual to string/*D*-brane theory – as is generally believed to be the case – they must support domain walls (of the *D*-brane type) [10], and we know, they do [11, 12]. A *D*-brane is defined as a hypersurface on which a string may end. In field theory both the brane and the string arise as BPS solitons, the brane as a domain wall and the string as a flux tube. If their properties reflect those inherent to string theory, at least to an extent, the flux tube must end on the wall. Moreover, the wall must house gauge fields living on its world volume, under which the end of the string is charged.

The purpose of this review is to summarize developments in critical solitons in two, three and four dimensions, with emphasis on four dimensions and on most recent results. A large variety of BPS-saturated solitons exist in four-dimensional field theories: domain walls, flux tubes (strings), monopoles and dyons, and various junctions of the above objects. A list of recent discoveries includes localization of gauge fields on domain walls, non-Abelian strings that can end on domain walls, developed boojums, confined monopoles attached to strings, and other remarkable findings. The BPS nature of these objects allows one to obtain a number of exact results. In many instances nontrivial dynamics of the bulk theories we will consider lead to effective low-energy theories in the world volumes of domain walls and strings (they are related to zero modes) exhibiting novel dynamical features that are interesting by themselves.

We do not try to review the vast literature accumulated since the mid 1990s in its entirety. A comparison with a huge country the exploration of which is not yet completed is in order here. Instead, we suggest what may be called "travel diaries" of the participants of the exploratory expedition. Recent publications [13, 14, 15, 16, 17] facilitate our task since they present the current developments in this field from a complementary point of view.

The "diaries" are organized in two parts. The first part (entitled "Short excursion") is a bird's eye view of the territory. It gives a brief and largely nontechnical introduction to basic ideas lying behind supersymmetric solitons and particular applications. It is designed in such a way as to present a general perspective that would be understandable to anyone with an elementary knowledge in classical and quantum fields, and supersymmetry.

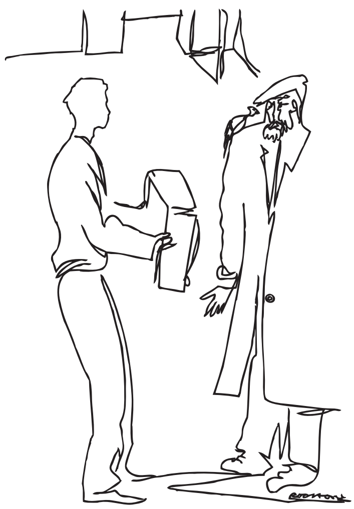
Here we present some historic remarks, catalog relevant centrally extended superalgebras and review basic building blocks we consistently deal with – domain walls, flux tubes, and monopoles – in their classic form. The word "classic" is used here not in the meaning "before quantization" but, rather, in the meaning "recognized and cherished in the community for years."

The second part (entitled "Long journey") is built on other principles. It is intended for those who would like to delve in this subject thoroughly, with its specific methods and technical devices. We put special emphasis on recent developments having direct relevance to QCD and gauge theories at large, such as non-Abelian flux tubes (strings), non-Abelian monopoles confined on these strings, gauge field localization on domain walls, etc. We start from presenting our benchmark model, which has extended $\mathcal{N} = 2$ supersymmetry. Here we go well beyond conceptual foundations, investing efforts in detailed discussions of particular problems and aspects of our choosing. Naturally, we choose those problems and aspects which are instrumental in the novel phenomena mentioned above. In addition to walls, strings and monopoles, we also dwell on the string-wall junctions which play a special role in the context of dualization.

Our subsequent logic is from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ and further on. Indeed, in certain instances we are able to descend to non-supersymmetric gauge theories which are very close relatives of QCD. In particular, we present a fully controllable weakly coupled model of the Meissner effect which exhibits quite nontrivial (strongly coupled) dynamics on the string world sheet. One can draw direct parallels between this consideration and the issue of k-strings in QCD.

Part I

Short excursion



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Central charges in superalgebras

In this Section we will briefly review general issues related to central charges (CC) in superalgebras.

2.1 History

The first superalgebra in four-dimensional field theory was derived by Golfand and Likhtman [18] in the form

$$\{\bar{Q}_{\dot{\alpha}} Q_\beta\} = 2P_\mu \left(\sigma^\mu\right)_{\alpha\beta}, \quad \{\bar{Q}_\alpha \bar{Q}_\beta\} = \{Q_\alpha Q_\beta\} = 0, \tag{2.1.1}$$

i.e. with no central charges. Possible occurrence of CC (elements of superalgebra commuting with all other operators) was first mentioned in an unpublished paper of Lopuszanski and Sohnius [19] where the last two anticommutators were modified as

$$\{Q^I_\alpha Q^G_\beta\} = Z^{IG}_{\alpha\beta}. \tag{2.1.2}$$

The superscripts I, G mark extended supersymmetry. A more complete description of superalgebras with CC in quantum field theory was worked out in [20]. The only central charges analyzed in this paper were Lorentz scalars (in four dimensions), $Z_{\alpha\beta} \sim \varepsilon_{\alpha\beta}$. Thus, by construction, they could be relevant only to extended supersymmetries.

A few years later, Witten and Olive [1] showed that in supersymmetric theories with solitons, central extension of superalgebras is typical; topological quantum numbers play the role of central charges.

It was generally understood that superalgebras with (Lorentz-scalar) central charges can be obtained from superalgebras without central charges in higher-dimensional space-time by interpreting some of the extra components of the momentum as CC's (see e.g. [21]). When one compactifies extra dimensions one obtains an extended supersymmetry; the extra components of the momentum act as scalar central charges.

Algebraic analysis extending that of [20] carried out in the early 1980s (see e.g. [22]) indicated that the super-Poincaré algebra admits CC's of a more general form, but the dynamical role of additional tensorial charges was not recognized until much later. Now it is common knowledge that central charges that originate from operators other than the energy-momentum operator in higher dimensions can play a crucial role. These tensorial central charges take non-vanishing values on extended objects such as strings and membranes.

Central charges that are antisymmetric tensors in various dimensions were introduced (in the supergravity context, in the presence of p-branes) in Ref. [23] (see also [24, 25]). These CC's are relevant to extended objects of the domain wall type (membranes). Their occurrence in four-dimensional super-Yang–Mills theory (as a quantum anomaly) was first observed in [11]. A general theory of central extensions of superalgebras in three and four dimensions was discussed in Ref. [26]. It is worth noting that those central charges that have the Lorentz structure of Lorentz vectors were not considered in [26]. The gap was closed in [27].

2.2 Minimal supersymmetry

The minimal number of supercharges ν_Q in various dimensions is given in Table 2.1. Two-dimensional theories with a single supercharge, although algebraically possible, are quite exotic. In "conventional" models in $D = 2$ with local interactions the minimal number of supercharges is two.

The minimal number of supercharges in Table 2.1 is given for a real representation. Then, it is clear that, generally speaking, the maximal possible number of CC's is determined by the dimension of the symmetric matrix $\{Q_i Q_j\}$ of the size $\nu_Q \times \nu_Q$, namely,

$$\nu_{\mathrm{CC}} = \frac{\nu_Q(\nu_Q + 1)}{2}. \qquad (2.2.1)$$

In fact, D anticommutators have the Lorentz structure of the energy-momentum operator P_μ. Therefore, up to D central charges could be absorbed in P_μ, generally speaking. In particular situations this number can be smaller, since although algebraically the corresponding CC's have the same structure as P_μ, they are dynamically distinguishable. The point is that P_μ is uniquely defined through the conserved and symmetric energy-momentum tensor of the theory.

Additional dynamical and symmetry constraints can further diminish the number of independent central charges, see e.g. Section 2.2.1.

The total set of CC's can be arranged by classifying CC's with respect to their Lorentz structure. Below we will present this classification for $D = 2, 3$ and 4, with

Table 2.1. *The minimal number of supercharges, the complex dimension of the spinorial representation and the number of additional conditions (i.e. the Majorana and/or Weyl conditions).*

D	2	3	4	5	6	7	8	9	10
ν_Q	(1*) 2	2	4	8	8	8	16	16	16
$\text{Dim}(\psi)_C$	2	2	4	4	8	8	16	16	32
# cond.	2	1	1	0	1	1	1	1	2

special emphasis on the four-dimensional case. In Section 2.3 we will deal with $\mathcal{N} = 2$ superalgebras.

2.2.1 $D = 2$

Consider two-dimensional non-chiral theories with two supercharges. From the discussion above, on purely algebraic grounds, three CC's are possible: one Lorentz-scalar and a two-component vector,

$$\{Q_\alpha,\ Q_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}(P_\mu + Z_\mu) + i(\gamma^5\gamma_0)_{\alpha\beta}Z\,. \tag{2.2.2}$$

We refer to Appendix A for our conventions regarding gamma matrices. $Z^\mu \neq 0$ would require existence of a vector order parameter taking distinct values in different vacua. Indeed, if this central charge existed, its current would have the form

$$\zeta_\nu^{\ \mu} = \varepsilon_{\nu\rho}\,\partial^\rho A^\mu, \quad Z^\mu = \int \zeta_0^{\ \mu}\, dz,$$

where A^μ is the above-mentioned order parameter. However, $\langle A^\mu\rangle \neq 0$ will break Lorentz invariance and supersymmetry of the vacuum state. This option will not be considered. Limiting ourselves to supersymmetric vacua we conclude that a single (real) Lorentz-scalar central charge Z is possible in $\mathcal{N} = 1$ theories. This central charge is saturated by kinks.

2.2.2 $D = 3$

The central charge allowed in this case is a Lorentz-vector Z_μ, i.e.

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}(P_\mu + Z_\mu). \tag{2.2.3}$$

One should arrange Z_μ to be orthogonal to P_μ. In fact, this is the scalar central charge of Section 2.2.1 elevated by one dimension. Its topological current can be written as

$$\zeta_{\mu\nu} = \varepsilon_{\mu\nu\rho}\, \partial^\rho A, \quad Z_\mu = \int d^2x\, \zeta_{\mu 0}. \tag{2.2.4}$$

By an appropriate choice of the reference frame Z_μ can always be reduced to a real number times (0, 0, 1). This central charge is associated with a domain line oriented along the second axis.

Although from the general relation (2.2.3) it is pretty clear why BPS vortices cannot appear in theories with two supercharges, it is instructive to discuss this question from a slightly different standpoint. Vortices in three-dimensional theories are localized objects, particles (BPS vortices in $2+1$ dimensions were previously considered in [28]; see also references therein). The number of broken translational generators is d, where d is the soliton's co-dimension, $d = 2$ in the case at hand. Then *at least* d supercharges are broken. Since we have only two supercharges in the problem at hand, both must be broken. This simple argument tells us that for a 1/2-BPS vortex the minimal matching between bosonic and fermionic zero modes in the (super) translational sector is one-to-one.

Consider now a putative BPS vortex in a theory with minimal $\mathcal{N} = 1$ supersymmetry (SUSY) in $2+1$D. Such a configuration would require a world volume description with two bosonic zero modes, but only one fermionic mode. This is not permitted by the argument above, and indeed no configurations of this type are known. Vortices always exhibit at least two fermionic zero modes and can be BPS-saturated only in $\mathcal{N} = 2$ theories.

2.2.3 $D = 4$

Maximally one can have 10 CC's which are decomposed into Lorentz representations as $(0, 1) + (1, 0) + (1/2, 1/2)$:

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2(\gamma^\mu)_{\alpha\dot\alpha}(P_\mu + Z_\mu), \tag{2.2.5}$$

$$\{Q_\alpha, Q_\beta\} = (\Sigma^{\mu\nu})_{\alpha\beta} Z_{[\mu\nu]}, \tag{2.2.6}$$

$$\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = (\bar{\Sigma}^{\mu\nu})_{\dot\alpha\dot\beta} \bar{Z}_{[\mu\nu]}, \tag{2.2.7}$$

where $(\Sigma^{\mu\nu})_{\alpha\beta} = (\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha}_\beta$ is a chiral version of $\sigma^{\mu\nu}$ (see e.g. [29]). The antisymmetric tensors $Z_{[\mu\nu]}$ and $\bar{Z}_{[\mu\nu]}$ are associated with domain walls, and reduce to a complex number and a spatial vector orthogonal to the domain wall. The (1/2, 1/2) CC Z_μ is a Lorentz vector orthogonal to P_μ. It is associated with strings (flux tubes), and reduces to one real number and a three-dimensional unit spatial vector parallel to the string.

2.3 Extended SUSY

In four dimensions one can extend superalgebra up to $\mathcal{N} = 4$, which corresponds to sixteen supercharges. Reducing this to lower dimensions we get a rich variety of extended superalgebras in $D = 3$ and 2. In fact, in two dimensions the Lorentz invariance provides a much weaker constraint than in higher dimensions, and one can consider a wider set of (p, q) superalgebras comprising $p + q = 4$, 8, or 16 supercharges. We will not pursue a general solution; instead, we will limit our task to (i) analysis of central charges in $\mathcal{N} = 2$ in four dimensions; (ii) reduction of the minimal SUSY algebra in $D = 4$ to $D = 2$ and 3, namely the $\mathcal{N} = 2$ SUSY algebra in those dimensions. Thus, in two dimensions we will consider only the non-chiral $\mathcal{N} = (2, 2)$ case. As should be clear from the discussion above, in the dimensional reduction the maximal number of CC's stays intact. What changes is the decomposition in Lorentz and R-symmetry irreducible representations.

2.3.1 $\mathcal{N} = 2$ in $D = 2$

Let us focus on the non-chiral $\mathcal{N} = (2, 2)$ case corresponding to dimensional reduction of the $\mathcal{N} = 1$, $D = 4$ algebra. The tensorial decomposition is as follows:

$$\{Q^I_\alpha, Q^J_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}\left[(P_\mu + Z_\mu)\delta^{IJ} + Z^{(IJ)}_\mu\right] + 2i\,(\gamma^5\gamma^0)_{\alpha\beta}\, Z^{\{IJ\}} + 2i\,\gamma^0_{\alpha\beta} Z^{[IJ]}, \quad I, J = 1, 2. \tag{2.3.1}$$

Here $Z^{[IJ]}$ is antisymmetric in I, J; $Z^{\{IJ\}}$ is symmetric while $Z^{(IJ)}$ is symmetric and traceless. We can discard all vectorial central charges Z^{IJ}_μ for the same reasons as in Section 2.2.1. Then we are left with two Lorentz singlets $Z^{(IJ)}$, which represent the reduction of the domain wall charges in $D = 4$ and two Lorentz singlets $\mathrm{Tr} Z^{\{IJ\}}$ and $Z^{[IJ]}$, arising from P_2 and the vortex charge in $D = 3$ (see Section 2.3.2). These central charges are saturated by kinks.

Summarizing, the (2, 2) superalgebra in $D = 2$ is

$$\{Q^I_\alpha, Q^J_\beta\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}\, P_\mu\, \delta^{IJ} + 2i(\gamma^5\gamma^0)_{\alpha\beta}\, Z^{\{IJ\}} + 2i\,\gamma^0_{\alpha\beta} Z^{[IJ]}. \tag{2.3.2}$$

It is instructive to rewrite Eq. (2.3.2) in terms of complex supercharges Q_α and $Q^\dagger_\beta$ corresponding to four-dimensional $Q_\alpha,\ \bar{Q}_{\dot\alpha}$, see Section 2.2.3. Then

$$\left\{Q_\alpha, Q^\dagger_\beta\right\}(\gamma^0)_{\beta\gamma} = 2\left[P_\mu\gamma^\mu + Z\,\frac{1-\gamma_5}{2} + Z^\dagger\,\frac{1+\gamma_5}{2}\right]_{\alpha\gamma}, \tag{2.3.3}$$

$$\left\{Q_\alpha, Q_\beta\right\}(\gamma^0)_{\beta\gamma} = -2Z'\,(\gamma_5)_{\alpha\gamma}, \quad \left\{Q^\dagger_\alpha, Q^\dagger_\beta\right\}(\gamma^0)_{\beta\gamma} = 2Z'^\dagger\,(\gamma_5)_{\alpha\gamma}.$$

The algebra contains two complex central charges, Z and Z'. In terms of components $Q_\alpha = (Q_R, Q_L)$ the nonvanishing anticommutators are

$$\begin{aligned} &\{Q_L, Q_L^\dagger\} = 2(H+P), \quad &&\{Q_R, Q_R^\dagger\} = 2(H-P), \\ &\{Q_L, Q_R^\dagger\} = 2iZ, &&\{Q_R, Q_L^\dagger\} = -2iZ^\dagger, \\ &\{Q_L, Q_R\} = 2iZ', &&\{Q_R^\dagger, Q_L^\dagger\} = -2iZ'^\dagger. \end{aligned} \tag{2.3.4}$$

It exhibits the automorphism $Q_R \leftrightarrow Q_R^\dagger$, $Z \leftrightarrow Z'$ associated [30] with the transition to a mirror representation [31]. The complex central charges Z and Z' can be readily expressed in terms of real $Z^{\{IJ\}}$ and $Z^{[IJ]}$,

$$Z = Z^{[12]} + \frac{i}{2}\left(Z^{\{11\}} + Z^{\{22\}}\right), \quad Z' = \frac{Z^{\{12\}} + Z^{\{21\}}}{2} - i\,\frac{Z^{\{11\}} - Z^{\{22\}}}{2}. \tag{2.3.5}$$

Typically, in a given model either Z or Z' vanish. A practically important example to which we will repeatedly turn below (e.g. Sections 3.5 and 4.5.3) is provided by the so-called twisted-mass-deformed CP($N-1$) model [32]. The central charge Z emerges in this model at the classical level. At the quantum level it acquires additional anomalous terms [33, 34]. Both $Z \neq 0$ and $Z' \neq 0$ simultaneously in a contrived model [33] in which the Lorentz symmetry and a part of supersymmetry are spontaneously broken.

2.3.2 $\mathcal{N} = 2$ in $D = 3$

The superalgebra can be decomposed into Lorentz and R-symmetry tensorial structures as follows:

$$\{Q_\alpha^I, Q_\beta^J\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}[(P_\mu + Z_\mu)\delta^{IJ} + Z_\mu^{(IJ)}] + 2i\,\gamma^0_{\alpha\beta} Z^{[IJ]}, \tag{2.3.6}$$

where all central charges above are real. The maximal set of 10 CC's enter as a triplet of spacetime vectors Z_μ^{IJ} and a singlet $Z^{[IJ]}$. The singlet CC is associated with vortices (or lumps), and corresponds to the reduction of the (1/2,1/2) charge or the 4th component of the momentum vector in $D = 4$. The triplet Z_μ^{IJ} is decomposed into an R-symmetry singlet Z_μ, algebraically indistinguishable from the momentum, and a traceless symmetric combination $Z_\mu^{(IJ)}$. The former is equivalent to the vectorial charge in the $\mathcal{N} = 1$ algebra, while $Z_\mu^{(IJ)}$ can be reduced to a complex number and vectors specifying the orientation. We see that these are the direct reduction of the (0,1) and (1,0) wall charges in $D = 4$. They are saturated by domain lines.