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The propagation of a disturbance in relation to imaging

1.1 Background and motivation

In the study of the Earth one encounters a wide array of physical processes. Examples which quickly come to mind are transient pressure variations associated with fluid flow, the propagation of elastic and electromagnetic waves, inelastic deformation, reactive chemical transport, and multiphase flow. This diversity rivals that of any other discipline. On the surface, these phenomena may seem unrelated, and it is difficult to envision techniques, other than numerical modeling, which could prove useful across such a wide range of processes.

Our understanding of the complex systems that comprise our planet is hampered by a difficulty characteristic of the Earth Sciences; our observations are, for the most part, remote and indirect. That is, measurements are typically gathered at the Earth's surface or from a relatively small number of boreholes penetrating the depths. Thus, the details of the physical system are shielded from view by the Earth itself. To be sure, great advances have been made in geophysical imaging and in a sense it is possible to 'see' within the Earth. However, as compared to a physics laboratory, a completely controlled field experiment is the exception rather than the rule in the geosciences. This necessitates tackling what is known as the inverse problem, the complement to the modeling of a natural system. In the inverse problem, remote observations are used to infer properties, usually model parameters, describing the system. For example, seismic amplitude and arrival time changes gathered in time-lapse monitoring are used to infer saturation and pressure changes in a reservoir due to fluid production. Increasingly, other fields such as medicine and non-destructive testing are adopting imaging and inversion for non-invasive evaluation.

In large measure this book is concerned with modeling techniques developed with the inverse problems in mind. To this end we shall be concerned with methods that place a premium on efficiency, flexibility, simplicity, and physical intuition. For example, finite-difference techniques, extremely versatile but computationally intensive methods that often do not scale well with problem size, will not play a large part in our discussion. Rather, we shall emphasize trajectory-based techniques perhaps best known for their use in medical and geophysical tomographic imaging. Though we shall be dealing with a methodology which is known and successful in solving particular classes of imaging problems, our setting will be much more general. That is, we shall use the approach not just to study high-frequency, wave-like behavior, but also in the study of diffusive and mixed propagation. Typically, diffusive behavior, such as a pressure change, is treated as having little in common with propagating high-frequency elastic or electromagnetic waves. This distinction, while valid in some sense, ignores the many traits that these transient phenomena share (Virieux et al., 1994). Their common characteristics are particularly useful in solving the inverse problem, for example, estimating properties such as permeability or hydraulic diffusivity within the Earth. We will also consider non-linear processes, such as multiphase flow. In that sense, parts of our book might be considered as a follow-up to texts concerned with the extension of ray methods to non-linear problems (Whitham, 1974; Anile et al., 1993; Maslov and Omel'yanov, 2001). Our emphasis will be on techniques that work in the face of the significant heterogeneity found within the Earth and most other natural systems.

Our intent is to provide methods which are applicable across a range of disciplines. One goal is to provide some degree of unity across the various specialties such as geochemistry, hydrogeology, geophysics, and reservoir engineering. A common framework and a shared methodology is particularly important with the emergence of fluid flow monitoring and time-lapse imaging. Due to the increasing expense of identifying and exploiting petroleum and geothermal reservoirs, and the increasing importance of water resources, there is less room for failed wells and inefficient extraction. It is necessary to take full advantage of large geophysical data sets in understanding fluid flow at depth. With the increasing role of unconventional hydrocarbon resources, it is important to understand the interaction between hydraulic fractures and natural fractures in the subsurface. The trajectory-based methods in this book are useful in both geophysical and flow-related modeling. The efficiency of this approach and its favorable scaling properties mean that it is useful in treating large data sets and large models. The visual and intuitive nature of the methods make them useful for interpreting observations. Also, the rapid turn-around time for an inversion means that the techniques are appropriate for time-lapse monitoring. The approaches described in this book transcend applications in the Earth sciences. In particular, the techniques and governing equations for dispersive, dissipative, and non-linear propagation occur in other areas, such as non-destructive testing and engineering, as well as in medical imaging.

The trajectory-based solutions that we shall describe provide insight into the physics of propagating disturbances and introduce flexibility into the modeling.

For example, the trajectories facilitate visualization of the movement of material and energy in complex systems. Furthermore, the formulation of a solution defined along a trajectory typically partitions into two distinct problems: a propagation time problem and another calculation involving the evolution of the amplitude. As we shall see, this partitioning provides flexibility, particularly in the treatment of the inverse problem. Thus, one may use the travel time of a disturbance as a basis for imaging an object. This leads to the idea of travel time tomography for a wide range of processes and the images that it can provide. As we shall see, the relationship between travel times and the properties of an object, such as the porosity and permeability, is often relatively direct and simple. This contrasts with the more complicated dependence of the amplitude of the disturbance on the internal structure of an object. Intuitively, the amplitudes depend upon the behavior of all trajectories in a given neighborhood, particularly how they diverge or converge. Thus, the amplitudes will not only depend upon the properties along a given trajectory, but also upon how the properties vary with distance away from the trajectory.

In the remainder of this chapter we shall discuss applications of a few of these ideas in a somewhat intuitive manner. We will examine a transient solution to the diffusion equation and how one might define a propagation time. The relationship between the propagation time and the properties of the medium will be described for a simple homogeneous medium. We will discuss the computation of a travel time for multiphase flow. Finally, a variety of applications will be noted to illustrate the power and utility of the trajectory based methods.

1.2 A propogating disturbance

At the most basic level we are concerned with the propagation of a disturbance or a change in an observable quantity, such as fluid pressure or elastic strain in a solid. Ultimately, we wish to relate the characteristics of the disturbance, such as its arrival time and amplitude to the properties of the medium through which it propagates. Some of the other processes of interest are the advective transport of a conservative or reactive tracer, multiphase flow, reactive chemical transport, electromagnetic processes, and heat flow. Typically, the disturbance is man made, as due to the injection or extraction of a volume of fluid, giving rise to transient phenomena, with propagation from a source to an observation point. One simple mathematical representation of a one-dimensional disturbance that maintains its shape as it propagates is

$$u(x,t) = f(x - ct) \tag{1.1}$$

where the profile of the disturbance at $t = 0$ is $f(x)$. The disturbance propagates to the right, in the positive x direction, with velocity c. The reader may consider the argument $\eta = x - ct$ to be a time dependent translation of the x-axis, where the rate of translation is specified by c.

We present a simple derivation of the governing equation here. A differential equation with (1.1) as a solution follows from differentiating $u(x, t)$ with respect to x and t. First, define the variable η representing the argument of f:

$$\eta = x - ct. \tag{1.2}$$

Then, differentiating $u(x, t)$ with respect to x gives, upon using the chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial \eta}{\partial x}\frac{df}{d\eta} = \frac{df}{d\eta} \tag{1.3}$$

while the time derivative is

$$\frac{\partial u}{\partial t} = \frac{\partial \eta}{\partial t}\frac{df}{d\eta} = -c\frac{df}{d\eta} \tag{1.4}$$

and so one may infer from these two equations that $u(x, t)$ satisfies

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{1.5}$$

Equation (1.5) is the simplest possible linear wave equation (Whitham, 1974, p. 6), describing propagation through a uniform medium. Equation (1.5) describes one-way wave propagation, that is movement in only one direction. In many cases it is also possible to have movement in the reverse direction, leading to the classical wave equation which is a second-order partial differential equation (Whitham, 1974, p. 6).

Most physical processes cause the disturbance to change shape as it propagates and translational invariance is not maintained. For example, there is dispersion in which the propagation velocity depends upon the slope of the waveform, which is quite commonly observed. Then there is dissipation in which the attenuation of a propagating pulse depends upon its amplitude or perhaps its curvature. This behavior is characteristic of diffusive phenomena. Thus, features in a waveform are preferentially decreased in amplitude at a rate that is proportional to the height or possibly the curvature, leading to the ‘smoothing’ of a pulse as it propagates. Finally, the effects of dispersion and dissipation, present even when a medium behaves linearly, can be confounded by the effects of non-linear behavior. Non-linearity can lead to a sharpening or steepening of a disturbance, think of a breaking wave, counteracting the effects of dispersion and dissipation (Whitham, 1974). We shall touch on some of these effects in this chapter and in the chapters that follow.

1.3 An example involving dissipation

A large class of physical processes, from heat flow and fluid pressure diffusion, to electromagnetic wave propagation (Virieux et al., 1994), involve some form of dissipation. As an example, consider the change in fluid pressure $p(\mathbf{x}, t)$ as a function of the spatial coordinates $\mathbf{x}$ and time t, associated with the injection of a slightly compressible fluid, governed by the diffusion equation

$$\nabla \cdot \left(\frac{k}{\mu} \nabla p \right) = S \frac{\partial p}{\partial t} + f \tag{1.6}$$

where $k(\mathbf{x})$ is the intrinsic or absolute permeability, μ is the fluid viscosity, $S(\mathbf{x})$ is the specific storage coefficient, and $f(\mathbf{x}, t)$ is the fluid source or sink (de Marsily, 1986). Equation (1.6) follows directly from mass conservation and the relationship between fluid flow velocity $\mathbf{q}$ and the fluid pressure gradient

$$\mathbf{q} = \frac{k}{\mu} \nabla p \tag{1.7}$$

first derived by Henri Darcy in a study of the fountains of Dijon, France (de Marsily, 1986, p. 58). Equation (1.7) simply states that the flow velocity $\mathbf{q}$ is proportional to the pressure gradient with the proportionality constant k/μ. Darcy's law (1.7) is akin to Fouriers equation (Fourier, 1822) describing the conduction of heat in a solid, later used by Fick (1855) to develop a quantitative basis for diffusion (Crank, 1975, p. 2). As we shall see in Chapter 2, the governing equation for the evolution of pressure, Equation (1.6), follows from the equation for the conservation of the fluid mass. Substituting Darcy's law, Equation (1.7), into the mass conservation equation results in Equation (1.6) if the porous medium is assumed to behave in a linear elastic fashion.

Let us consider a numerical simulation as an example: the injection of fluid at a constant rate into a well intersecting a heterogeneous formation (Figure 1.1). As fluid is pumped into the well indicated by the bulls-eye in the figure, the pressure increases in the surrounding formation. The pressure increase migrates away from the well in a diffusive fashion, leading to a gradual change in pressure throughout the formation. In response to the sudden injection or withdrawal of fluid at the well, as with the initiation of pumping, the fluid pressure will evolve over time according to Equation (1.6). The pressure change will propagate out into the reservoir in a diffusive manner, as determined by the spatially varying coefficients $S(\mathbf{x})$ and $k(\mathbf{x})$ of Equation (1.6). If one examines snapshots of the pressure field at various times, there is no clear pressure 'front' per se, just a gradual change in pressure over time (Figure 1.2). The calculation of the evolving pressure, given the source term $f(\mathbf{x}, t)$ and the medium properties $S(\mathbf{x})$ and $k(\mathbf{x})$, constitutes the forward problem and requires the solution of Equation (1.6). Solving the forward problem, calculating

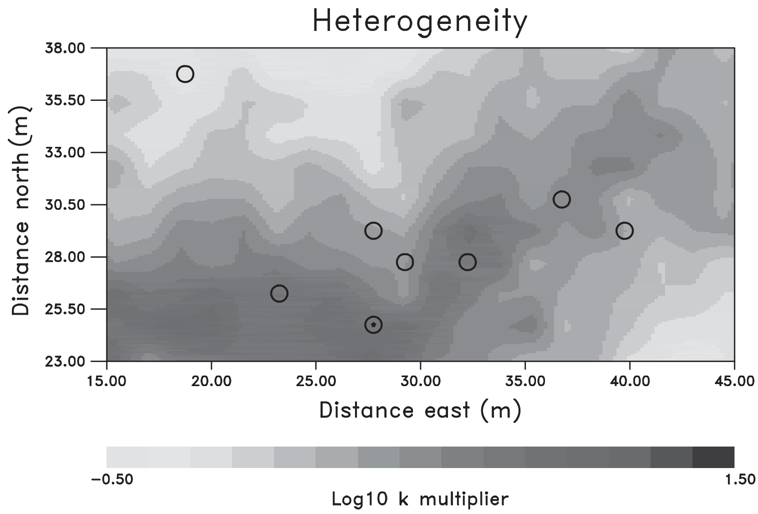

Figure 1.1 Permeability variation used in modeling a pumping experiment. The injection well is indicated by the bulls-eye.

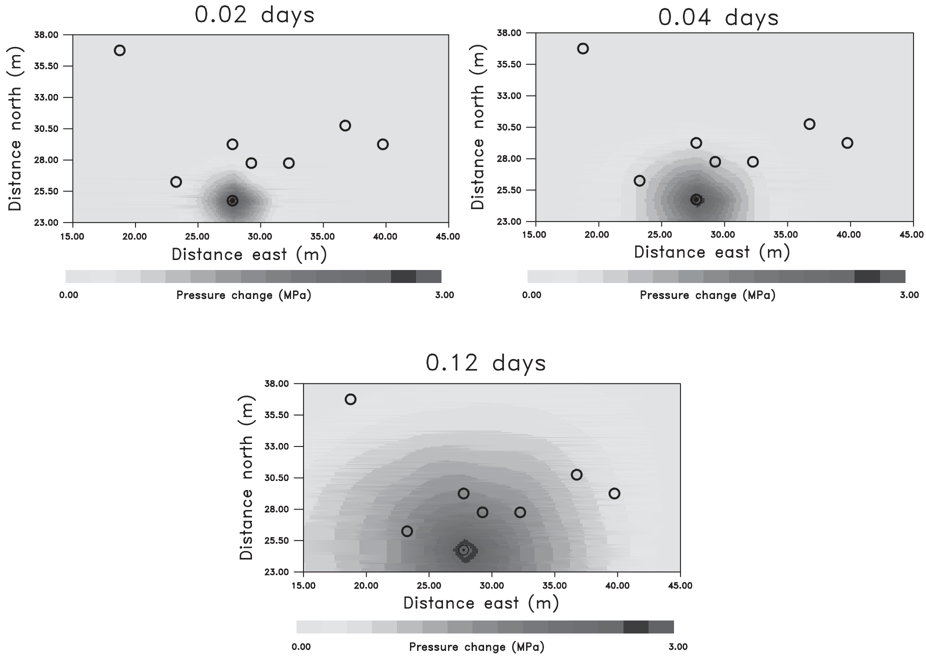

Figure 1.2 Normalized change in fluid pressure at various times due to injection.

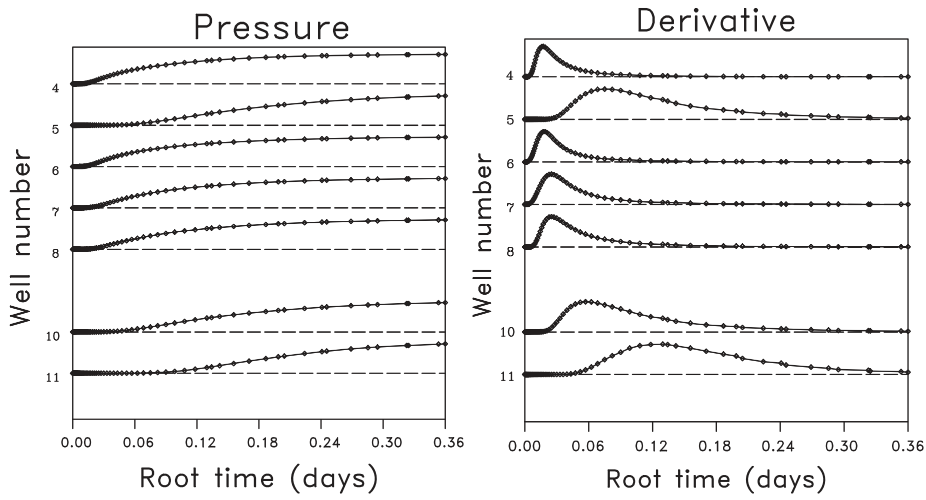

Figure 1.3 The change in fluid pressure (a) and its time derivative and (b) for the seven observing wells.

the pressure $p(\mathbf{x}, t)$ given the flow properties at all points on a simulation grid, is a stable and well-posed task, usually accomplished using a numerical technique such as a finite-difference or a finite-element routine (Peaceman, 1977; Pruess et al., 1999; Datta-Gupta and King, 2007). Figure 1.2 displays the pressure changes over time, as calculated by a integral finite-difference numerical simulator (Pruess et al., 1999), for a medium with the spatially varying permeability indicated in Figure 1.1. The pressure changes resulting at the seven observation points, denoted by open circles in Figures 1.1 and 1.2, are shown in Figure 1.3.

The pressure changes associated with a step function source do not display well-defined onsets. Rather, the pressure increases monotonically with time and it is not possible to define an 'arrival time' for the pressure disturbance at each station. It is also difficult to define an 'amplitude' because the pressure increases gradually over time and no clear maximum is attained. The situation improves if we consider the time-derivative of pressure rather than the pressure itself. Consider the pressure derivatives with respect to time, shown in Figure 1.3, for each of the seven observation wells. The pressure time-derivative has a greater resemblance to a propagating pulse, with a well-defined peak, denoting the maximum rate of pressure change. The peak can be used to define an arrival time and an amplitude for the propagating pulse at each observation point. The time-derivative of the pressure, normalized by its amplitude, more closely resembles a propagating wave (Figure 1.4). The fact that the time-derivative of the pressure variation is

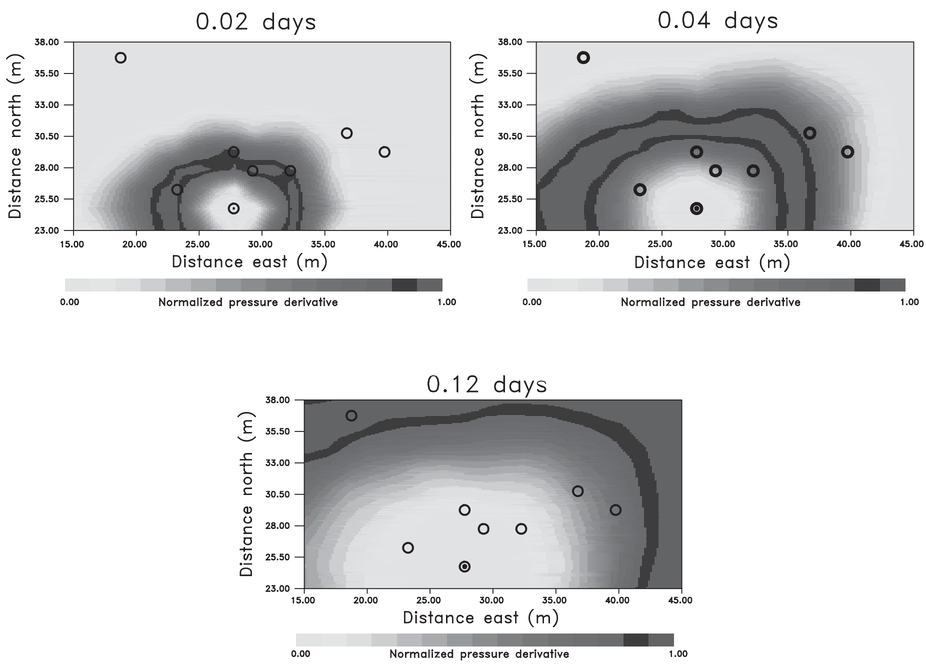

Figure 1.4 Normalized pressure derivative field at various times due to fluid injection.

pulse-like in nature makes physical sense. Because the source is a step-like function, as required to propagate pressure changes away from the source well, its time-derivative is an impulsive function. The time-derivative maps the pressure variation back into the equivalent of an impulse response. The resulting pulse-like feature is such that we can measure the amplitude of the change and we can define a time at which the rate of change is a maximum – the arrival time of the disturbance.

Motivated by the numerical simulation, let us return to the pressure Equation (1.6) and to the notion of a propagating pulse. We can produce one of the simplest solutions if we assume a homogeneous medium with an intrinsic permeability k_0, viscosity μ_0, and a specific storage coefficient S_0. In Chapter 4 we shall consider propagation in a fully heterogeneous medium. For a homogeneous medium without a source or sink, we may write Equation (1.6) as

$$D\nabla \cdot \nabla p = \frac{\partial p}{\partial t} \tag{1.8}$$

where $D = k_0/\mu_0 S_0$ is the diffusion coefficient (Crank, 1975, p. 11). Equation (1.8) implies that the rate-of-change of p at a point is proportional to the spatial curvature of the pressure field. Thus, sharp spatial variations will decay rapidly, leading to

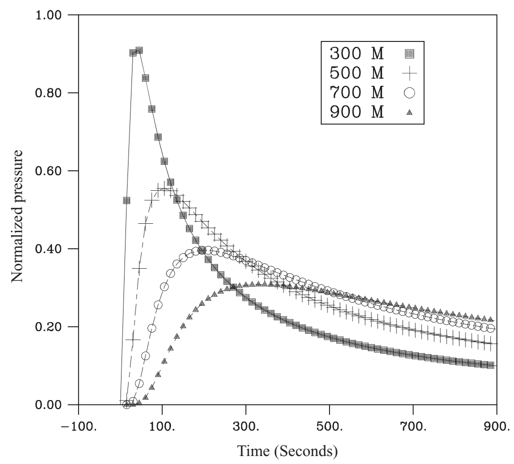

Figure 1.5 Pressure variation $p(r,t) = \frac{C}{\sqrt{t^3}} \exp\left(-\frac{1}{4D}\frac{r^2}{t}\right)$, associated with an impulsive source corresponding to four observation points located at successively greater distances from the source.

a smoothing of the pressure field with time. For an impulsive or delta-function source one may solve Equation (1.8) using either the Fourier transform (Bracewell, 2000) or separation of variables (Miller, 1977). The solution is (Crank, 1975, p. 11; de Marsily, 1986, p. 162)

$$p(r,t) = \frac{C}{\sqrt{t^3}} \exp\left(-\frac{1}{4D}\frac{r^2}{t}\right) \tag{1.9}$$

where r is the radial distance of the observation point from the injection location, C is an arbitrary constant which is determined from the initial amplitude of the pressure pulse source. In Figure 1.5, the pressure variation at four different distances are plotted as a function of time.

We can dig a little deeper in an attempt to better define what we mean by the arrival time of a propagating pressure pulse. Notice that we have switched from the step-like source, used in the numerical simulation, to an impulsive, or delta-function like source. For the pressure Equation (1.8) this is not a significant change because, as a consequence of its linearity, we can easily switch between these two source types by either differentiation or integration. The most identifiable features of the curves in Figure 1.5 are the locations of the peaks in the pressure variations. The condition for a pressure peak is given by the vanishing of the time derivative for a particular observation point r_o:

$$\frac{\partial p}{\partial t} = \left[-\frac{3}{2t} + \frac{r_o^2}{4Dt^2}\right] \frac{C}{\sqrt{t^3}} \exp\left(-\frac{1}{4D}\frac{r_o^2}{t}\right) = 0. \tag{1.10}$$

For finite values of t and a non-vanishing pressure variation, the exponential term is non-zero and condition (1.10) is satisfied when the expression in the square brackets vanishes, leading to

$$t_o = \frac{r_o^2}{6D}, \tag{1.11}$$

where t_o signifies the time at which the peak pressure is observed at the point r_o. Equation (1.11) reveals that the arrival time of the peak pressure at r_o is determined by the quantity

$$D = \frac{k_0}{\mu_0 S_0},$$

providing a relationship between an observable quantity and the properties of the medium and the fluid. We can turn Equation (1.11) around and use the observed arrival time, t_o, at the location r_o to infer the value of D for a particular medium, thus solving the simplest of inverse problems.

As indicated earlier, a transient solution of the diffusion equation can be viewed as a propagating disturbance. This brings us to one of the central points of this book – we can interpret both conventional wave-like processes, such as elastic and high-frequency electromagnetic propagation and diffusive phenomena using a common framework. This is useful because many real-world physical systems do not fall neatly into either category and can display aspects of both. Another point, demonstrated in Chapter 4, is that we can use trajectory-based methods to develop semi-analytic solutions and to model such processes. Such solutions complement purely numerical approaches by providing insight, both from visualization and from the semi-analytic expressions themselves. Furthermore, the trajectory-based solutions provide additional flexibility by subdividing the modeling into a travel time calculation and an amplitude calculation. We shall have much more to say about these ideas in the chapters that follow.

1.4 A non-linear example

Linear governing equations, as exemplified by the diffusion Equation (1.6), can be used to model important physical processes, but they are by no means the entire story. In fact, one would not get very far in modeling fluid flow under the restriction of linearity. Compressible flow, multiphase flow, and reactive transport are but a few examples of non-linear behavior. Trajectory-based modeling techniques are equally applicable to physical processes described by non-linear equations. In fact, some of the earliest applications of the method of characteristics were to problems involving compressible gas dynamics (Courant and Friedrichs, 1948). In this section we will