Cambridge University Press
978-0-521-51528-3 - W. T. Koiter's Elastic Stability of Solids and Structures
Edited by Arnold M. A. van der Heijden
Excerpt
More information

1

Stability

1.1 Discrete systems

Consider a system with a finite number of degrees of freedom. The position of this system is represented by a position vector $\mathbf{q}(q^1, q^2 \ldots q^n)$, where q^i $(i \in 1 \ldots n)$ are n independent coordinates. It is assumed that the system is *holonomic*, i.e., no relations exist between the derivatives of the coordinates, and *scleronomic*, i.e., the factor time is not explicitly needed in the description of the system.[†] Let $\dot{q}^i$ be the generalized velocities. The kinetic energy T is then a homogeneous quadratic function of the generalized velocities, and hence T can be written as

$$T = \frac{1}{2} a_{ij}(\mathbf{q}) \dot{q}^i \dot{q}^j \,. \tag{1.1.1}$$

When the system is non-scleronomic, terms linear in the velocities and a term independent of $\dot{q}^i$ must be added. The coefficient $a_{ij}(\mathbf{q})$ is called the *inertia matrix*. The forces acting upon the system can be expressed by a generalized force vector $\mathbf{Q}$ defined by

$$Q_i \delta q^i = \text{v.w.}, \tag{1.1.2}$$

where the right-hand side stands for the virtual work of all the forces acting upon the system. In general, this expression is not a total differential. However, for an important class of problems, it is. Systems for which 1.1.2) is a total differential are called *conservative systems*. In that case we have

$$Q_i \delta q^i = -\delta P(\mathbf{q}) \,, \tag{1.1.3}$$

where $\delta P(\mathbf{q})$ is a total differential and $P(\mathbf{q})$ is called the potential energy. In the following, we mainly restrict our attention to conservative systems because for elastic systems, conservative forces play an important role.

Introducing a kinetic potential L defined by

$$\mathcal{L} = T - P, \tag{1.1.4}$$

[†] This implies that $d\mathbf{q} = \mathbf{q},_k \, dq^k$.

the Lagrangian equations for a conservative system are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0, \quad (\mathrm{i} \in 1, \ldots, \mathrm{n}). \tag{1.1.5}$$

Using the expression (1.1.1), we may rewrite this equation to yield

$$\frac{d}{dt}(a_{ij}\dot{q}^i) - \frac{1}{2}a_{hk,i}\dot{q}^h\dot{q}^k + P_{,i} = 0, \tag{1.1.6}$$

where

$$a_{hk,i} = \frac{\partial a_{hk}(\mathbf{q})}{\partial q^i}, \quad P_{,i} = \frac{\partial P(\mathbf{q})}{\partial q^i}$$

However, it often happens that non-conservative forces are present (e.g., damping forces). It is then advantageous to take these into account separately as follows:

$$Q_i\delta q^i = \delta P(\mathbf{q}) + Q_i^*\delta q^i, \tag{1.1.7}$$

where $\mathbf{Q}^*$ is the vector of non-conservative forces. The equations of motion then read

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = Q_i^*, \quad (\mathrm{i} \in 1 \cdots \mathrm{n}). \tag{1.1.8}$$

These are n second-order ordinary differential equations.

Let us now consider the stability of discrete systems. For a system to be in equilibrium, the velocities (and hence the kinetic energy) have to vanish. This implies that for a conservative system, we have

$$P_{,i} = 0. \tag{1.1.9}$$

In words: The potential energy has a *stationary* value.

By *stability* we mean that a small disturbance from the state of equilibrium does not cause large deviations from this state of equilibrium. A disturbance from the state of equilibrium implies that the velocities are nonzero or that the position differs from the equilibrium position. We can always choose our coordinate system such that the equilibrium position is given by $\mathbf{q} = \mathbf{0}$. Furthermore, we can always choose the potential energy in such a way that it vanishes in the equilibrium position. Doing so, we may write

$$P = \frac{1}{2}P_{,ij}(\mathbf{0})\,q^iq^j + \cdots. \tag{1.1.10}$$

To be able to give a more exact definition of stability, we need a measure to denote the deviation from the state of equilibrium. Remembering that in equilibrium we have $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$, a number $\rho(\mathbf{q}, \dot{\mathbf{q}})$ is introduced with the following properties:

$$\begin{aligned}
&1) \quad \rho(\mathbf{q}, \dot{\mathbf{q}}) \leq 0 \quad \text{for} \quad \mathbf{q} \neq \mathbf{0} \quad \text{or} \quad \dot{\mathbf{q}} \neq \mathbf{0},\\
&2) \quad \rho(\mathbf{q}_1, \mathbf{q}_2, \dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) \leq \rho(\mathbf{q}_1, \dot{\mathbf{q}}_1)\,\rho(\mathbf{q}_2, \dot{\mathbf{q}}_2)\\
&\qquad \text{(triangle inequality)},\\
&3) \quad \rho(\alpha\mathbf{q}, \alpha\dot{\mathbf{q}}) = |\alpha|\,\rho(\mathbf{q}, \dot{\mathbf{q}}) \quad \alpha \in \mathbb{R}.
\end{aligned} \tag{1.1.11}$$

Cambridge University Press
978-0-521-51528-3 - W. T. Koiter's Elastic Stability of Solids and Structures
Edited by Arnold M. A. van der Heijden
Excerpt
More information

We are now in a position to define the following stability criterion.

An equilibrium position is *stable* if and only if for *each* positive number ε there exists a positive number $\delta(\varepsilon)$ such that for *all* disturbances of the equilibrium at the time $t > 0$, with $\rho\,[\mathbf{q}(0), \dot{\mathbf{q}}(0)] < \delta$, the motion for $t > 0$ satisfies $\rho\,[\mathbf{q}(t), \dot{\mathbf{q}}(t)] < \varepsilon$.

Notice that the statement about stability depends on the measure that is used. Different measures yield different criteria for stability. Notice further that different measures may be used for $t = 0$ and $t > 0$. This freedom is of great importance for applications. For example, suitable choices for ρ are

$$\rho = \left[\sum_{i=1}^{n} (q^i)^2 + \sum_{i=1}^{n} (\dot{q}^i)^2\right]^{1/2},$$

$$\rho = \max\left|q^i\right| + \max\left|\dot{q}^i\right|.$$

For a conservative system, $T + P =$ constant. This well-known result can easily be derived from Lagrange's equations for a conservative, holonomic, and scleronomic system. Multiplying the equations by $\dot{q}^i$, we obtain

$$\dot{q}^i \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \dot{q}^i \frac{\partial \mathcal{L}}{\partial q^i} = 0$$

or

$$\frac{d}{dt}\left(\dot{q}^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \ddot{q}^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \dot{q}^i \frac{\partial \mathcal{L}}{\partial q^i} = 0$$

or

$$\frac{d}{dt}\left(\dot{q}^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \frac{d}{dt}\mathcal{L} = 0.$$

Using Euler's theorem for homogeneous quadratic functions, we readily obtain

$$\frac{d}{dt}(2T) - \frac{d}{dt}(T - P) = 0,$$

from which follows

$$T + P = E, \tag{1.1.12}$$

where $E = T(t = 0) + P(t = 0)$. This equation enables us to make the following statement about stability.

Theorem. *The equilibrium is stable provided the potential energy is positive-definite.*

To prove this theorem, we introduce the following norms:

$$\|\mathbf{q}\|^2 = \sum_{i=1}^{n} (q^i)^2$$

$$\|\dot{\mathbf{q}}\|^2 = \sum_{i=1}^{n} (\dot{q}^i)^2.$$

Let $d(c)$ denote the minimum of $P(\mathbf{q})$ on the hypersphere $\| \mathbf{q} \| = c$. $P(\mathbf{q})$ is positive-definite when $d(c)$ is a monotonically increasing function of c on the sphere $\delta \le c < R$.

Proof. $T + P =$ constant $= E$, T is positive or zero, and P is positive-definite. Restrict the initial disturbance so that

$$\| \mathbf{q}\,(0) \| < c_1 \quad \text{and} \quad E < d\,(c_1)\,.$$

This means that $T\,(t = 0) < d\,(c_1)$. Because $T + P < d\,(c_1)$ and P is positive-definite, it follows that $T < d\,(c_1)$ for all t. On the other hand, because T is positive or zero, it follows that $P < d\,(c_1)$ for all t. A similar argument holds for a disturbance $\| \dot{\mathbf{q}}(0) \| < c_2$. Hence, we may choose an arbitrary (small) disturbance and the displacements and velocities will always remain within definable bounds. ■

The converse of this theorem has not yet been proven in all generality. To see some of the difficulties that are encountered, we consider the following example (one degree of freedom):

$$P(q) = e^{-q^{-2}} \cos q^{-2}.$$

For $q = 0$, all the derivatives vanish. However, in the immediate vicinity of the origin there are always negative values of P. In spite of this, the system is stable for sufficiently small disturbances.

Actual physical systems are never exactly conservative, i.e., there is always some dissipation. The approximation by a conservative system is often a very good approximation. In the presence of damping forces, we need the Lagrangian equations with an additional term for the non-conservative forces. Multiplying by $\dot{\mathbf{q}}^i$,

$$\dot{q}^i \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \dot{q}^i \frac{\partial \mathcal{L}}{\partial q^i} = Q_i^* \dot{q}^i, \quad (\mathrm{i} \in 1, \ldots, \mathrm{n})$$

from which follows

$$\frac{d}{dt}\,(T + P) = Q_i^* \dot{q}^i \equiv -D\,(\mathbf{q}, \dot{\mathbf{q}})\,. \tag{1.1.13}$$

Damping implies that the dissipation function $D > 0$ for $\dot{\mathbf{q}} \neq 0$. We now make the following assumptions:

1) The damping forces have the property that $Q_i^* \to 0$ for $\|\dot{\mathbf{q}}\| \to 0$.
2) $D(\mathbf{q}, \dot{\mathbf{q}}^i) > 0$ for $\|\dot{\mathbf{q}}\| \neq 0$.
3) $P\,(\mathbf{q})$ does not possess stationary values for $\| \dot{\mathbf{q}} \| < c$ except at $\mathbf{q} = \mathbf{0}$.

Systems satisfying these conditions are called *pseudo-conservative.* Notice that because of restriction (1), dry friction forces are excluded.

Theorem. *In the presence of (positive) damping forces, a system with an indefinite potential energy is unstable.*

Cambridge University Press
978-0-521-51528-3 - W. T. Koiter's Elastic Stability of Solids and Structures
Edited by Arnold M. A. van der Heijden
Excerpt
More information

Proof. If P is indefinite, consider a disturbance of the equilibrium configuration with zero velocity and negative potential energy. The initial total energy is thus negative and, as this configuration cannot be in equilibrium, motion must result, as a result of which energy is dissipated. The total energy must decrease, so the system cannot stay in the vicinity of the origin, which means that the equilibrium configuration is unstable. ■

The great advantage of this stability theorem is that it does not involve the kinetic energy, and hence the inertia matrix $a_{ij}(\mathbf{q})$. For a conservative or pseudo-conservative system, the stability criterion only depends on the potential energy (a quasi-static criterion). In general, the stability problem is a dynamic problem, and the kinetic energy plays an essential role. An example of such a problem is the behavior of the wings of an airplane in an airflow. In this case, the forces do not depend on only the geometry but also on the velocities.

For static loads, it is often sufficient to restrict oneself to conservative loads (e.g., deadweight loads). A more severe restriction for continuous systems is that we must restrict ourselves to elastic systems, i.e., to systems where there is a potential for the internal energy. Such a potential does not exist when plasticity occurs.

Let us now have a closer look at the stability problem. As mentioned previously, the stability criterion is fully determined by the potential energy $P(\mathbf{q})$. In the equilibrium position, we have chosen $P(\mathbf{0}) = 0$ and $\mathbf{q} = \mathbf{0}$ so that we may write

$$P(\mathbf{q}) = \frac{1}{2} P,_{ij}(\mathbf{0})\, q^i q^j + \cdots \equiv \frac{1}{2} c_{ij} q^i q^j + \cdots, \tag{1.1.14}$$

where c_{ij} denotes the stiffness matrix in the equilibrium position. It follows that when the stiffness matrix is positive-definite, $P(\mathbf{q})$ is positive-definite and the system is stable. If c_{ij} is indefinite (or negative-definite) then the system is unstable. If c_{ij} is semi-definite-positive (i.e., non-negative and zero for at least a deflection in one direction), then we must consider higher-order terms in the expansion for P. This case is called a *critical case of neutral equilibrium*. We shall consider this case in more detail.

It is convenient to transform the quadratic form (1.1.14) to a sum of quadratic terms. If the form is positive-definite, then the coefficients in the transformed form are all positive. Applying this transform to (1.1.14) and denoting the transformed coordinates again by q^i, we may write

$$P(\mathbf{q}) = \frac{1}{2} \sum_{i=1}^{n} c_i (q^i)^2 + \cdots + ()\,(q^1)^3 + \cdots. \tag{1.1.15}$$

Further, we order the coefficients c_i such that

$$c_1 \le c_2 \le c_3 \le \ \cdots \ \le c_n.$$

We now consider the case $c_1 = 0, c_2 > 0$. Taking all $q^i = 0\ (i > 1)$, the dominant term will be $(q^1)^3$. This term can attain negative values, and hence the system will be unstable. A necessary condition for stability is that the coefficient of $(q^1)^3$ is equal

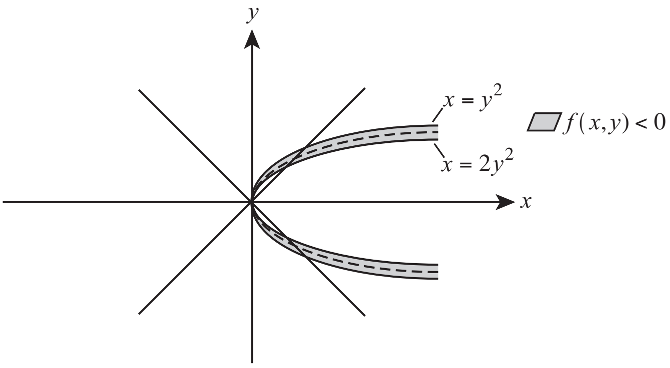

Figure 1.1.1

to zero. A further necessary condition for stability is that the coefficient of $(q^1)^4$ is positive. However, this condition is insufficient, as will be shown in the following example. Consider the function

$$P = f(x, y) = (x - y^2)(x - 2y^2) = x^2 - 3xy^2 + 2y^4. \tag{1.1.16}$$

The graphs of the functions $x - y^2 = 0$ and $x - 2y^2 = 0$ are given in Fig. 1.1.1.

The function $f(x, y)$ in an arbitrary small neighborhood of the origin takes on both positive and negative values. In this case, the quadratic form in y vanishes at the origin, and there is no cubic term, but the coefficient in the quartic term is positive. Hence, the necessary conditions for stability are satisfied. However, this system is unstable because in an arbitrarily small neighborhood of the origin, P takes on negative values.

The reason that the conditions mentioned here are not sufficient is that we have restricted our investigation to straight lines through the origin (see Fig. 1.1.1). Following these straight lines, we always find only positive values in a sufficiently small neighborhood of the origin. However, if we follow curved lines through the origin (see the dashed lines), we easily find negative values. Once we have recognized the reason why the conditions imposed are insufficient, it is easy to find a remedy. To this end, we consider a line $y =$ constant in the neighborhood of the origin, and we minimize $f(x, y)$ with respect to x, i.e.,

$$\operatorname*{Min}_{y=y_1} f(x, y) = x^2 - 3xy_1^2 + 2y_1^4. \tag{1.1.17}$$

This yields $2x - 3y_1^2 = 0$, and hence $x = 3/2\, y_1^2$. Substitution of this value into $f(x, y)$ yields $\min f(x, y) = -1/4\, y_1^4$, which means that the function is indefinite.

In general, the function P is minimized with respect to q^i $(i > 1)$ for fixed q^1. When the coefficient of $(q^1)^4$ is positive-definite, the system is stable.

Cambridge University Press
978-0-521-51528-3 - W. T. Koiter's Elastic Stability of Solids and Structures
Edited by Arnold M. A. van der Heijden
Excerpt
More information

2

Continuous Elastic Systems

2.1 Thermodynamic background

Consider a body that is in a state of equilibrium under conservative loads. Our aim is to investigate this equilibrium state.

For an *elastic* body, the internal energy per unit mass may be represented by $U(s, \boldsymbol{\gamma})$, where s denotes the specific entropy and $\boldsymbol{\gamma}$ is the deformation tensor. Let x_i $(i = 1, 2, 3)$ be the components of the position vector $\mathbf{x}$, which describe the position of a material point in the "fundamental state" I, which is to be investigated. Let $\mathbf{u}(\mathbf{x})$ be the displacement vector from the fundamental state ($\mathbf{u}$ is a small but finite displacement). The corresponding position in the "adjacent state" II is then $\mathbf{x} + \mathbf{u}$. The (additional) deformation tensor is now defined by

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}u_{h,i}u_{h,j}\,. \tag{2.1.1}$$

The fact that the body has undergone deformations to arrive in the fundamental state is unimportant because the state I is kept fixed.

The temperature T is now defined by

$$T = \frac{\partial U}{\partial s} \tag{2.1.2}$$

($\boldsymbol{\gamma}$ is kept constant).

From (2.1.2) we obtain

$$\frac{\partial T}{\partial s} = \frac{\partial^2 U}{\partial s^2}.$$

The specific heat of the material is now defined by

$$T\frac{\partial s}{\partial T} = C_\gamma, \tag{2.1.3}$$

where $C_\gamma > 0$ for a thermodynamically stable material. As $\partial^2 U/\partial s^2$ is positive (nonzero) we may solve (2.1.2) for s, which yields $s = s(T, \boldsymbol{\gamma})$.

We now introduce the function $F(T, \boldsymbol{\gamma})$, defined by

$$F(T, \boldsymbol{\gamma}) = U - Ts. \tag{2.1.4}$$

$F\,(T, \boldsymbol{\gamma})$ is called the *free energy*.

Writing (2.1.4) as a total differential, we find (for fixed $\boldsymbol{\gamma}$)

$$\frac{\partial F}{\partial T}\delta T = \frac{\partial U}{\partial s}\delta s - T\delta s - s\delta T. \tag{2.1.5}$$

Using (2.1.2) we find

$$s = -\frac{\partial F}{\partial T}. \tag{2.1.6}$$

Let us denote the temperature in the fundamental state (which by virtue of the equilibrium state is equal to the temperature of its surroundings) by T_{I}. A disturbance of the equilibrium state will cause a heat flux in the body. In the following, we will assume that the temperature of the surrounding medium is constant (T_{I}). Denoting the heat flux by $\mathbf{q}$, the heat flux per unit time through a closed surface is given by $\int_A \mathbf{q} \cdot \mathbf{n}\; dA$, where $\mathbf{n}$ denotes the unit normal vector on the surface, positive in the outward direction. According to the second law of thermodynamics,[†] vheat will flow out of the body when its surface temperature is higher than that of the surrounding medium, i.e.,

$$(T - T_{\mathrm{I}})\mathbf{q} \cdot \mathbf{n} \geq 0 \quad \text{(on the surface)}. \tag{2.1.7}$$

The heat flux will cause an entropy flux. The entropy flux vector $\mathbf{h}$ is given by

$$\mathbf{h} = \frac{1}{T}\mathbf{q} \quad \text{(per unit time and per unit area)}. \tag{2.1.8}$$

For an arbitrary part of the body, the entropy balance is given by

$$\int\limits_V \rho\dot{s}\; dV = \int\limits_A \mathbf{h} \cdot \mathbf{n}\; dA, \tag{2.1.9}$$

where ρ is the specific mass. This equation only holds in the absence of irreversible processes in the body. When the state of the body also depends on the deformation rates, irreversible processes will occur, which implies entropy production. In that case, the entropy balance reads

$$\int\limits_V \rho\dot{s}\; dV = \int\limits_A \mathbf{h} \cdot \mathbf{n}\; dA + \int\limits_V \rho\sigma\; dV, \quad \sigma \geq 0, \tag{2.1.10}$$

where σ denotes the entropy production per unit time and mass. This is the more general formulation of the second law of thermodynamics (CLAUSIUS-DUHEM). The first law of thermodynamics states that the total amount of heat that flows into a body is transformed into internal energy.

Let $P_L\,[\mathbf{u}\,(\mathbf{x}\,(t))]$ be the potential energy of the external loads and let

$$K\,[\dot{\mathbf{u}}\,(\mathbf{x}\,(t))] = \frac{1}{2}\int_V \rho\dot{\mathbf{u}} \cdot \dot{\mathbf{u}}\; dV$$

[†] This is an early formulation by CLAUSIUS (1854).

Cambridge University Press
978-0-521-51528-3 - W. T. Koiter's Elastic Stability of Solids and Structures
Edited by Arnold M. A. van der Heijden
Excerpt
More information

be the kinetic energy. The total energy balance is then given by

$$\frac{d}{dt}\left\{\int_V \rho U(s,\boldsymbol{\gamma})\ dV + K\left[\dot{\mathbf{u}}\left(\mathbf{x}\left(t\right)\right)\right] + P_L\left[\mathbf{u}\left(\mathbf{x}\left(t\right)\right)\right]\right\} = -\int_A \mathbf{q}\left(\mathbf{x}\left(t\right)\right)\cdot\mathbf{n}\, dA, \quad \text{(1st law)} \tag{2.1.11}$$

where we have a negative sign on the right-hand side of this equation because the heat flux is regarded as positive in the outward direction.

To draw conclusions from the first and the second laws, we subtract (2.1.11) from (2.1.10) multiplied by T_{I}. This yields

$$\frac{d}{dt}\left\{\int_V \rho\left[U(s,\boldsymbol{\gamma}) - T_{\mathrm{I}}s\right]\ dV + K\left[\dot{\mathbf{u}}\left(\mathbf{x}\left(t\right)\right)\right] + P_L\left[\mathbf{u}\left(\mathbf{x}\left(t\right)\right)\right]\right\} = \int_V \left(\frac{T_{\mathrm{I}}}{T} - 1\right)\mathbf{q}\cdot\mathbf{n}\, dA - T_{\mathrm{I}}\int_V \rho\sigma\, dV \le 0 \tag{2.1.12}$$

(DUHEM, 1911).
Here we have made use of the relation

$$\int_V \rho\dot{s}\, dV = \frac{d}{dt}\int_V \rho s\, dV \tag{2.1.13}$$

The first term on the right-hand side of (2.1.12) is negative because the heat flux is in the outward direction when $T > T_{\mathrm{I}}$, and the second term is negative because the entropy production is always positive. The integral on the left-hand side of (2.1.12) may be expressed in terms of the free energy. Using the relation

$$U(s,\boldsymbol{\gamma}) - T_{\mathrm{I}}s = U(s,\boldsymbol{\gamma}) - Ts + (T - T_{\mathrm{I}})\, s = F\left(T,\boldsymbol{\gamma}\right) + (T - T_{\mathrm{I}})\,\frac{\partial F}{\partial T}, \tag{2.1.14}$$

we obtain

$$\frac{d}{dt}\left\{\int_V \rho\left[F\left(T,\boldsymbol{\gamma}\right) + (T_{\mathrm{I}} - T)\,\frac{\partial F}{\partial T}\right]\ dV + P_L + K\right\} \le 0. \tag{2.1.15}$$

DUHEM (1911) already discussed the stability of a system on the basis of this equation and came to the conclusion that a system is stable when the form between the braces is positive-definite. In this form, K is a positive-definite function. However, the terms between the square brackets depend on the deformation tensor and the temperature, whereas P_L depends on the displacement field. The problem is to separate the influence of the temperature and the displacement field. A straightforward expansion

$$F\left(T,\boldsymbol{\gamma}\right) = F\left(T_{\mathrm{I}},\boldsymbol{\gamma}\right) + \left(\frac{\partial F}{\partial T}\right)_{T_{\mathrm{I}}}(T - T_{\mathrm{I}}) + \frac{1}{2}\left(\frac{\partial^2 F}{\partial T^2}\right)_{T_{\mathrm{I}}}(T - T_{\mathrm{I}})^2 + \cdots$$

Cambridge University Press
978-0-521-51528-3 - W. T. Koiter's Elastic Stability of Solids and Structures
Edited by Arnold M. A. van der Heijden
Excerpt
More information

does not solve the problem. Following Ericksen (1965), we may write the Taylor expansion of the free energy at constant deformation $\boldsymbol{\gamma}$ in the form

$$F\left(T_{\mathrm{I}}, \boldsymbol{\gamma}\right) = F\left(T, \boldsymbol{\gamma}\right) + \left(\frac{\partial F}{\partial T}\right)_T \left(T_{\mathrm{I}} - T\right) + \frac{1}{2}\left(\frac{\partial^2 F}{\partial T^2}\right)_* \left(T_{\mathrm{I}} - T\right)^2, \tag{2.1.16}$$

where the first derivative is evaluated at the deformation $\boldsymbol{\gamma}$ and temperature T, and the second (starred) derivative at the deformation $\boldsymbol{\gamma}$ and an intermediate temperature $T^* = T + \theta\left(T_{\mathrm{I}} - T\right)$, where $0 < \theta < 1$. Using (2.1.16) we may rewrite the term between the square brackets in (2.1.15) as follows:

$$\begin{aligned} F\left(T, \boldsymbol{\gamma}\right) + \left(T_{\mathrm{I}} - T\right)\frac{\partial F}{\partial T} &= F\left(T_{\mathrm{I}}, \boldsymbol{\gamma}\right) - \frac{1}{2}\left(\frac{\partial^2 F}{\partial T^2}\right)_* \left(T_{\mathrm{I}} - T\right)^2 \\ &= F\left(T_{\mathrm{I}}, \boldsymbol{\gamma}\right) + \frac{1}{2}\left(\frac{c_\gamma}{T}\right)_* \left(T_{\mathrm{I}} - T\right)^2, \end{aligned} \tag{2.1.17}$$

where we have used the relation

$$c_\gamma \equiv T\frac{\partial s}{\partial T} = T\frac{\partial^2 F}{\partial T^2}.$$

The first term on the right-hand side of (2.1.17) depends only on the displacement field. The second term is positive-definite. The energy balance may now be written in the form

$$\frac{d}{dt}\left\{\int_V \rho F\left(T_{\mathrm{I}}, \boldsymbol{\gamma}\right)\, dV + P_L\left[\mathbf{u}\left(\mathbf{x}\left(t\right)\right)\right] + K\left[\dot{\mathbf{u}}\left(\mathbf{x}\left(t\right)\right)\right] + \frac{1}{2}\int_V \rho\left(\frac{c_\gamma}{T}\right)_* \left(T_{\mathrm{I}} - T\right)^2 dV\right\} \leq 0. \tag{2.1.18}$$

The last two terms in the left-hand member are positive-definite, and the remaining terms depend only on the displacement field. Our energy balance is not affected when we subtract from the expression between the braces a time-independent quantity,

$$\int_V \rho F\left(T_{\mathrm{I}}, \mathbf{0}\right)\, dV.$$

Further, we introduce the notation

$$W\left(\boldsymbol{\gamma}\right) \equiv \rho\left[F\left(T_{\mathrm{I}}, \boldsymbol{\gamma}\right) - F\left(T_{\mathrm{I}}, \mathbf{0}\right)\right], \tag{2.1.19}$$

where $W\left(\boldsymbol{\gamma}\right)$ is the (additional) stored elastic energy in the isothermal (additional) deformation $\boldsymbol{\gamma}$ at constant temperature T_{I}, from the fundamental state I to the current state. The potential energy functional P is now defined by

$$P\left[\mathbf{u}\left(\mathbf{x}\left(t\right)\right)\right] = \int_V W\left(\boldsymbol{\gamma}\right) dV + P_L\left[\mathbf{u}\left(\mathbf{x}\left(t\right)\right)\right]. \tag{2.1.20}$$

In words: The potential energy is equal to the sum of the increase of the elastic energy for isothermal deformations and the potential energy of the external loads.