1

Hypercyclic and supercyclic operators

Introduction

The aim of this first chapter is twofold: to give a reasonably short, yet significant and hopefully appetizing, sample of the type of questions with which we will be concerned and also to introduce some definitions and prove some basic facts that will be used throughout the whole book.

Let X be a topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by $\mathfrak{L}(X)$ the set of all continuous linear operators on X. If $T \in \mathfrak{L}(X)$, the T-**orbit** of a vector $x \in X$ is the set

$$O(x, T) := \{T^n(x);\ n \in \mathbb{N}\}.$$

The operator T is said to be **hypercyclic** if there is some vector $x \in X$ such that $O(x, T)$ is dense in X. Such a vector x is said to be hypercyclic for T (or T-*hypercyclic*), and the set of all hypercyclic vectors for T is denoted by $HC(T)$. Similarly, T is said to be **supercyclic** if there exists a vector $x \in X$ whose **projective orbit**

$$\mathbb{K} \cdot O(x, T) := \{\lambda T^n(x);\ n \in \mathbb{N},\ \lambda \in \mathbb{K}\}$$

is dense in X; the set of all supercyclic vectors for T is denoted by $SC(T)$. Finally, we recall that T is said to be **cyclic** if there exists $x \in X$ such that

$$\mathbb{K}[T]x := \operatorname{span} O(x, T) = \{P(T)x;\ P \text{ polynomial}\}$$

is dense in X.

Of course, these notions make sense only if the space X is *separable*. Moreover, hypercyclicity turns out to be a purely infinite-dimensional phenomenon ([206]):

PROPOSITION 1.1 *There are no hypercyclic operators on a finite-dimensional space $X \neq \{0\}$.*

PROOF Suppose on the contrary that T is a hypercyclic operator on $\mathbb{K}^N$, $N \geq 1$. Pick $x \in HC(T)$ and observe that $(x, T(x), \dots, T^{N-1}(x))$ is a linearly independent family and hence is a basis of $\mathbb{K}^N$. Indeed, otherwise the linear span of $O(x, T)$ would have dimension less than N and hence could not be dense in $\mathbb{K}^N$. For any $\alpha \in \mathbb{R}_+$, one can find a sequence of integers (n_k) such that $T^{n_k}(x) \to \alpha x$. Then $T^{n_k}(T^i x) = T^i(T^{n_k} x) \to \alpha T^i x$ for each $i < N$, and hence $T^{n_k}(z) \to \alpha z$ for any $z \in \mathbb{K}^N$. It follows that $\det(T^{n_k}) \to \alpha^N$, i.e. $\det(T)^{n_k} \to \alpha^N$. Thus, putting $a := |\det(T)|$, we see that the set $\{a^n;\ n \in \mathbb{N}\}$ is dense in $\mathbb{R}_+$. This is clearly impossible. □

The most general setting for linear dynamics is that of an arbitrary (separable) topological vector space X. However, we will usually assume that X is an F**-space**, i.e. a complete and metrizable topological vector space. Then X has a translation-invariant compatible metric (see [210]) and (X, d) is complete for any such metric d. In fact, in most cases X will be a **Fréchet space**, i.e. a locally convex F-space. Equivalently, a Fréchet space is a complete topological vector space whose topology is generated by a countable family of seminorms.

An attractive feature of F-spaces is that one can make use of the Baire category theorem. This will be *very* important for us. Incidentally, we note that the Banach–Steinhaus theorem and Banach's isomorphism theorem are valid in F-spaces, and if local convexity is added then one can also use the Hahn–Banach theorem and its consequences. If the reader feels uncomfortable with F-spaces and Fréchet spaces, he or she may safely assume that the underlying space X is a Banach space, keeping in mind that several natural examples live outside this context.

The chapter is organized as follows. We start by explaining how one can show that a given operator is hypercyclic or supercyclic. In particular, we prove the so-called *Hypercyclicity Criterion*, and the analogous Supercyclicity Criterion. Then we show that hypercyclicity and supercyclicity both entail certain spectral restrictions on the operator and its adjoint. Next, we discuss the "largeness" and the topological properties of the set of all hypercyclic vectors for a given operator T. Finally, we treat in some detail several specific examples: weighted shifts on ℓ^p spaces, composition operators on the Hardy space $H^2(\mathbb{D})$, and operators commuting with translations on the space of entire functions $H(\mathbb{C})$.

1.1 How to prove that an operator is hypercyclic

Our first characterization of hypercyclicity is a direct application of the Baire category theorem. This result was proved by G. D. Birkhoff in [53], and it is often referred to as **Birkhoff's transitivity theorem**.

THEOREM 1.2 (BIRKHOFF'S TRANSITIVITY THEOREM) *Let X be a separable F-space and let $T \in \mathfrak{L}(X)$. The following are equivalent:*

(i) *T is hypercyclic;*
(ii) *T is **topologically transitive**; that is, for each pair of non-empty open sets $(U, V) \subset X$ there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$.*

In that case, $HC(T)$ is a dense G_δ subset of X.

PROOF First, observe that if x is a hypercyclic vector for T then $O(x, T) \subset HC(T)$. Indeed, since X has no isolated points, any dense set $A \subset X$ remains dense after the removal of a finite number of points. Applying this to $A := O(x, T)$, and since $O(T^p(x), T) = O(x, T)\backslash\{x, T(x), \dots, T^{p-1}(x)\}$, we see that $T^p(x) \in HC(T)$ for every positive integer p. Thus $HC(T)$ is either empty or dense in X. From this, it is clear that (i) $\Longrightarrow$ (ii). Indeed, if (i) holds and the open sets U, V are given, we can pick $x \in U \cap HC(T)$ and then $n \in \mathbb{N}$ such that $T^n(x) \in V$.

To prove the converse, we note that since the space X is metrizable and separable, it is second-countable, i.e. it admits a countable basis of open sets. Let $(V_j)_{j\in\mathbb{N}}$ be such a basis. A vector $x \in X$ is hypercyclic for T iff its T-orbit visits each open set V_j, that is, iff for any $j \in \mathbb{N}$ there exists an integer $n \geq 0$ such that $T^n(x) \in V_j$. Thus one can describe $HC(T)$ as follows:

$$HC(T) = \bigcap_{j\in\mathbb{N}} \bigcup_{n\geq 0} T^{-n}(V_j).$$

This shows in particular that $HC(T)$ is a G_δ set. Moreover, it follows from the Baire category theorem that $HC(T)$ is dense in X iff each open set $W_j := \bigcup_{n\geq 0} T^{-n}(V_j)$ is dense; in other words, iff for each non-empty open set $U \subset X$ and any $j \in \mathbb{N}$ one can find n such that

$$U \cap T^{-n}(V_j) \neq \varnothing \quad \text{or, equivalently,} \quad T^n(U) \cap V_j \neq \varnothing.$$

Since (V_j) is a basis for the topology of X, this is equivalent to the topological transitivity of T. □

REMARK The implication (hypercyclic) $\Longrightarrow$ (topologically transitive) does not require the space X to be metrizable or Baire: it holds for an arbitrary topological vector space X. Indeed, the only thing we use is that $HC(T)$ is dense in X whenever it is non-empty, since X has no isolated points. Moreover, what is really needed for the converse implication (topologically transitive) $\Longrightarrow$ (hypercyclic) is that X is a Baire space with a countable basis of open sets. We also point out that Theorem 1.2 has nothing to do with linearity, since the definitions of hypercyclicity and topological transitivity do not require any linear structure. Accordingly, Theorem 1.2 holds as stated for an arbitrary continuous map $T : X \to X$ acting on some second-countable Baire space X without isolated points.

When the operator T is invertible, it is readily seen that T is topologically transitive iff T^{-1} is. Thus, we can state

COROLLARY 1.3 *Let X be a separable F-space, and let $T \in \mathfrak{L}(X)$. Assume that T is invertible. Then T is hypercyclic if and only if T^{-1} is hypercyclic.*

It is worth noting that T and T^{-1} do not necessarily share the same hypercyclic vectors; see Exercise 1.11.

We illustrate Theorem 1.2 with the following historic example, also due to Birkhoff [54].

EXAMPLE 1.4 (G. D. BIRKHOFF, 1929) Let $H(\mathbb{C})$ be the space of all entire functions on $\mathbb{C}$ endowed with the topology of uniform convergence on compact sets. For any non-zero complex number a, let $T_a : H(\mathbb{C}) \to H(\mathbb{C})$ be the **translation operator** defined by $T_a(f)(z) = f(z + a)$. Then T_a is hypercyclic on $H(\mathbb{C})$.

PROOF The space $H(\mathbb{C})$ is a separable Fréchet space, so it is enough to show that T_a is topologically transitive. If $u \in H(\mathbb{C})$ and $E \subset \mathbb{C}$ is compact, we set

$$\|u\|_E := \sup\{|u(z)|;\ z \in E\}.$$

Let U, V be two non-empty open subsets of $H(\mathbb{C})$. There exist $\varepsilon > 0$, two closed disks $K, L \subset \mathbb{C}$ and two functions $f, g \in H(\mathbb{C})$ such that

$$U \supset \big\{h \in H(\mathbb{C});\ \|h - f\|_K < \varepsilon\big\},$$

$$V \supset \big\{h \in H(\mathbb{C});\ \|h - g\|_L < \varepsilon\big\}.$$

Let n be any positive integer such that $K \cap (L + an) = \varnothing$. Since $\mathbb{C}\backslash(K \cup (L + an))$ is connected, one can find $h \in H(\mathbb{C})$ such that

$$\|h - f\|_K < \varepsilon \quad \text{and} \quad \|h - g(\cdot - na)\|_{L+an} < \varepsilon;$$

this follows from Runge's approximation theorem (see e.g. [209] or Appendix A). Thus $h \in U$ and $T_a^n(h) \in V$, which shows that T_a is topologically transitive. □

Topologically transitive maps are far from being exotic objects. For example, the map $x \mapsto 4x(1 - x)$ is transitive on the interval $[0, 1]$ and the map $\lambda \mapsto \lambda^2$ is transitive on the circle $\mathbb{T}$ (see e.g. R. L. Devaney's classical book [94]). However, in a topological setting one often needs a specific argument to show that a given map is transitive.

Nevertheless, in a linear setting an extremely useful general criterion for hypercyclicity does exist. This criterion was isolated by C. Kitai in a restricted form [158] and then by R. Gethner and J. H. Shapiro in a form close to that given below, [119]. The version we use appears in the Ph.D. thesis of J. Bès [45].

DEFINITION 1.5 *Let X be a topological vector space, and let $T \in \mathfrak{L}(X)$. We say that T satisfies the* ***Hypercyclicity Criterion*** *if there exist an increasing sequence of integers (n_k), two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and a sequence of maps $S_{n_k} : \mathcal{D}_2 \to X$ such that:*

(1) $T^{n_k}(x) \to 0$ *for any* $x \in \mathcal{D}_1$;
(2) $S_{n_k}(y) \to 0$ *for any* $y \in \mathcal{D}_2$;
(3) $T^{n_k} S_{n_k}(y) \to y$ *for each* $y \in \mathcal{D}_2$.

We will sometimes say that T satisfies the Hypercyclicity Criterion *with respect to the sequence* (n_k). When it is possible to take $n_k = k$ and $\mathcal{D}_1 = \mathcal{D}_2$, it is usually said that T satisfies **Kitai's Criterion**. We point out that in the above definition, the maps S_{n_k} are not assumed to be linear or continuous.

THEOREM 1.6 *Let $T \in \mathfrak{L}(X)$, where X is a separable F-space. Assume that T satisfies the Hypercyclicity Criterion. Then T is hypercyclic.*

FIRST PROOF We show that T is topologically transitive. Let U, V be two non-empty open subsets of X and pick $x \in \mathcal{D}_1 \cap U$, $y \in \mathcal{D}_2 \cap V$. Then $x + S_{n_k}(y) \to x \in U$ as $k \to \infty$ whereas $T^{n_k}(x + S_{n_k}(y)) = T^{n_k}(x) + T^{n_k} S_{n_k}(y) \to y \in V$. Thus, $T^{n_k}(U) \cap V \neq \varnothing$ if k is large enough. □

SECOND PROOF This second proof consists in replacing the Baire category theorem by a suitable series; this was the original idea of Kitai. We may assume that the set $\mathcal{D}_2$ is countable, and we enumerate it as a sequence $(y_l)_{l\in\mathbb{N}}$. Let us also fix some translation-invariant (necessarily complete) metric d for X. For the sake of visual clarity, we write $\|x\|$ instead of $d(x, 0)$.

We construct by induction a subsequence (m_k) of (n_k), a sequence $(x_k) \subset \mathcal{D}_1$, and a decreasing sequence of positive numbers (ε_k) with $\varepsilon_k \leq 2^{-k}$ such that the following properties hold for each $k \in \mathbb{N}$:

(i) $\|x_k\| < \varepsilon_k$;
(ii) $\|T^{m_k}(x_k) - y_k\| < \varepsilon_k$;
(iii) $\|T^{m_k}(x_i)\| < \varepsilon_k$ for all $i < k$;
(iv) if $u \in X$ satisfies $\|u\| < \varepsilon_k$ then $\|T^{m_i}(u)\| < 2^{-k}$ for all $i < k$.

Starting with $\varepsilon_0 := 1$, we use (2) and (3) of Definition 1.5 to find m_0 such that $\|S_{m_0}(y_0)\| < \varepsilon_0$ and $\|T^{m_0} S_{m_0}(y_0) - y_0\| < \varepsilon_0$ and then pick some $x_0 \in \mathcal{D}_1$ close to $S_{m_0}(y_0)$, in order to ensure (i) and (ii). The inductive step is likewise easy: having defined everything up to step k, one can first choose ε_{k+1} such that (iv) holds for $k+1$ and then m_{k+1} such that (iii) holds, $\|S_{m_{k+1}}(y_{k+1})\| < \varepsilon_{k+1}$ and $\|T^{m_{k+1}} S_{m_{k+1}}(y_{k+1}) - y_{k+1}\| < \varepsilon_{k+1}$, and after that $x_{k+1} \in \mathcal{D}_1$ close enough to $S_{m_{k+1}}(y_{k+1})$ to satisfy (i) and (ii).

By (i) and the completeness of (X, d), the series $\sum x_j$ is convergent in X. We claim that

$$x := \sum_{j=0}^{\infty} x_j$$

is a hypercyclic vector for T. Indeed, for any $l \in \mathbb{N}$ one may write

$$\begin{aligned}\|T^{m_l}(x) - y_l\| &\leq \sum_{j<l} \|T^{m_l}(x_j)\| + \|T^{m_l}(x_l) - y_l\| + \sum_{j>l} \|T^{m_l}(x_j)\| \\ &\leq l\varepsilon_l + \varepsilon_l + \sum_{j>l} 2^{-j}\end{aligned}$$

where we have used (iii), (ii), and (iv). Thus $T^{m_l}(x) - y_l \to 0$ as $l \to \infty$, which concludes the proof. □

REMARK 1.7 We have in fact proved the following more precise result: if $T \in \mathfrak{L}(X)$ satisfies the Hypercyclicity Criterion with respect to some sequence $(n_k)_{k\geq 0}$ then the family $(T^{n_k})_{k\geq 0}$ is *universal*, i.e. there exists some vector $x \in X$ such that the set $\{T^{n_k}(x);\ k \geq 0\}$ is dense in X. In fact, for *any* subsequence (n'_k) of (n_k), the family $(T^{n'_k})_{k\geq 0}$ is universal: this is apparent from the above proofs.

Theorem 1.6 will ensure the hypercyclicity of almost (!) all the hypercyclic operators in this book. We give two historical examples, due to G. R. MacLane [176] and to S. Rolewicz [206]. The latter was the first example of a hypercyclic operator that acts on a Banach space.

EXAMPLE 1.8 (G. R. MACLANE, 1951) The **derivative operator** $D : f \mapsto f'$ is hypercyclic on $H(\mathbb{C})$.

PROOF We apply the Hypercyclicity Criterion to the whole sequence of integers $(n_k) := (k)$, the *same* dense set $\mathcal{D}_1 = \mathcal{D}_2$ made up of all complex polynomials, and the maps $S_k := S^k$, where $Sf(z) = \int_0^z f(\xi)\, d\xi$. It is easy to check that conditions (1), (2) and (3) of Definition 1.5 are satisfied. Indeed, (1) holds because $D^k(P)$ tends to zero for any polynomial P, and (3) holds because $DS = I$ on $\mathcal{D}_2$. To prove (2), it is enough to check that $S_k(z^p) \to 0$ uniformly on compact subsets of $\mathbb{C}$, for any fixed $p \in \mathbb{N}$ (then we conclude using linearity). This, in turn, follows at once from the identity

$$S_k(z^p) = \frac{p!}{(p+k)!}\, z^{p+k}. \qquad \square$$

EXAMPLE 1.9 (S. ROLEWICZ, 1969) Let $B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the **backward shift** operator, defined by $B(x_0, x_1, \dots) = (x_1, x_2, \dots)$. Then λB is hypercyclic for any scalar λ such that $|\lambda| > 1$.

Observe that B itself cannot be hypercyclic since $\|B\| = 1$. Indeed, if T is a hypercyclic Banach space operator then $\|T\| > 1$ (otherwise any T-orbit would be bounded).

PROOF We apply the Hypercyclicity Criterion to the whole sequence of integers $(n_k) := (k)$, the dense set $\mathcal{D}_1 = \mathcal{D}_2 := c_{00}(\mathbb{N})$ made up of all finitely supported sequences and the maps $S_k := \lambda^{-k} S^k$, where S is the forward shift operator, $S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$. It is easy to check that conditions (1), (2) and (3) of Definition 1.5 are satisfied. For (1) and (3) the arguments are the same as in Example 1.8, and (2) follows immediately from the estimate $\|S_k\| \le |\lambda|^{-k}$. $\square$

As a consequence of Theorem 1.6 we get the following result, according to which an operator having a large supply of eigenvectors is hypercyclic. This is the so-called **Godefroy–Shapiro Criterion**, which was exhibited by G. Godefroy and J. H. Shapiro in [123].

COROLLARY 1.10 (GODEFROY–SHAPIRO CRITERION) *Let $T \in \mathfrak{L}(X)$ where X is a separable F-space. Suppose that $\bigcup_{|\lambda|<1} \operatorname{Ker}(T - \lambda)$ and $\bigcup_{|\lambda|>1} \operatorname{Ker}(T - \lambda)$ both span a dense subspace of X. Then T is hypercyclic.*

PROOF We show that T satisfies the Hypercyclicity Criterion with $(n_k) := (k)$ and

$$\mathcal{D}_1 := \operatorname{span}\left(\bigcup_{|\lambda|<1} \operatorname{Ker}(T-\lambda)\right), \qquad \mathcal{D}_2 := \operatorname{span}\left(\bigcup_{|\lambda|>1} \operatorname{Ker}(T-\lambda)\right).$$

The maps $S_k : \mathcal{D}_2 \to X$ are defined as follows: we set $S_k(y) := \lambda^{-k} y$ if $T(y) = \lambda y$ with $|\lambda| > 1$, and we extend S_k to $\mathcal{D}_2$ by linearity. This definition makes sense because the subspaces $\operatorname{Ker}(T-\lambda)$, $|\lambda| > 1$, are linearly independent. Thus, any non-zero $y \in \mathcal{D}_2$ may be uniquely written as $y = y_1 + \cdots + y_p$, with $y_i \in \operatorname{Ker}(T-\lambda_i) \setminus \{0\}$ and $|\lambda_i| > 1$. Having said that, it is clear that the assumptions of the Hypercyclicity Criterion are satisfied. □

REMARK In Corollary 1.10 we see for the first time that hypercyclicity can be inferred from the existence of a large supply of eigenvectors. This will be a recurrent theme in the book.

To illustrate the Godefroy–Shapiro Criterion, we are now going to establish the hypercyclicity of a certain classical operator defined on a Hilbert space of holomorphic functions. Let us first introduce some terminology. In what follows, $\mathbb{T}$ is the unit circle $\{z \in \mathbb{C};\ |z| = 1\}$ and $\mathbb{D}$ is the open unit disk $\{z \in \mathbb{C};\ |z| < 1\}$.

We denote by $H^2(\mathbb{D})$ the classical **Hardy space** on $\mathbb{D}$. By definition, $H^2(\mathbb{D})$ is the space of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^2}^2 := \sup_{r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty. \tag{1.1}$$

The Hardy space will appear several times in the book, and we assume that the reader is more or less familiar with it. Moreover, very few properties of $H^2(\mathbb{D})$ will be needed in our discussion (see Appendix B). We refer to e.g. [101] for an in-depth study of Hardy spaces.

We recall here that $H^2(\mathbb{D})$ can also be defined in terms of Taylor expansions. Any holomorphic function $f : \mathbb{D} \to \mathbb{C}$ can be (uniquely) written as $f(z) = \sum_0^\infty c_n(f) z^n$. Then f is in $H^2(\mathbb{D})$ if and only if $\sum_0^\infty |c_n(f)|^2 < \infty$, and in that case $\|f\|_{H^2}^2 = \sum_0^\infty |c_n(f)|^2$. This shows that $H^2(\mathbb{D})$ is canonically isometric to the sequence space $\ell^2(\mathbb{N})$ and also to the closed subspace of $L^2(\mathbb{T})$ defined by

$$H^2(\mathbb{T}) := \{\varphi \in L^2(\mathbb{T});\ \widehat{\varphi}(n) = 0 \ \text{ for all } n < 0\},$$

where the $\widehat{\varphi}(n)$ are the Fourier coefficients of φ. The function of $H^2(\mathbb{T})$ associated with a given $f \in H^2(\mathbb{D})$ is called the **boundary value** of f and will be denoted by f^*.

We summarize these elementary facts as follows: $H^2(\mathbb{D})$ is a Hilbert space whose norm can be defined in two equally useful ways:

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n(f)|^2 = \|f^*\|_{L^2(\mathbb{T})}^2.$$

Finally, we recall that convergence in $H^2(\mathbb{D})$ entails uniform convergence on compact sets. In particular the point evaluations $f \mapsto f(z)$ are continuous linear functionals on $H^2(\mathbb{D})$, so that for each $z \in \mathbb{D}$ there is a well-defined **reproducing kernel** k_z at z. By definition, k_z is the unique function in $H^2(\mathbb{D})$ satisfying

$$\forall f \in H^2(\mathbb{D})\ :\ f(z) = \langle f, k_z\rangle_{H^2}. \tag{1.2}$$

In the present case, k_z is given explicitly by the formula

$$k_z(s) = \frac{1}{1-\bar{z}s},$$

and (1.2) is just a rephrasing of Cauchy's formula.

The set of all bounded holomorphic functions of $\mathbb{D}$ will be denoted by $H^\infty(\mathbb{D})$. It is a non-separable Banach space when endowed with the norm

$$\|u\|_\infty = \sup\{|u(z)|;\ |z|<1\}.$$

If ϕ is a function in $H^\infty(\mathbb{D})$, the **multiplication operator** M_ϕ associated with ϕ is defined on $H^2(\mathbb{D})$ by $M_\phi(f) = \phi f$. From formula (1.1), it is readily seen that

$$\inf_{z\in\mathbb{D}} |\phi(z)| \times \|f\|_2 \leq \|M_\phi(f)\|_2 \leq \sup_{z\in\mathbb{D}} |\phi(z)| \times \|f\|_2$$

for any $f \in H^2(\mathbb{D})$. This shows in particular that M_ϕ is a bounded operator on $H^2(\mathbb{D})$, with $\|M_\phi\| \leq \|\phi\|_\infty$.

It is not hard to see that a multiplication operator M_ϕ cannot be hypercyclic (for example, for $f \in H^2(\mathbb{D})$, try to approximate $2f$ by functions of the form $\phi^n f$). As the next example shows, things are quite different for the adjoint operator M_ϕ^*.

EXAMPLE 1.11 Let $\phi \in H^\infty(\mathbb{D})$ and let $M_\phi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the associated multiplication operator. The adjoint multiplier M_ϕ^* is hypercyclic if and only if ϕ is non-constant and $\phi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$.

PROOF For any $z \in \mathbb{D}$, let $k_z \in H^2(\mathbb{D})$ be the reproducing kernel at z. Then k_z is an eigenvector of M_ϕ^*, with associated eigenvalue $\lambda(z) := \overline{\phi(z)}$. Indeed, we have

$$\langle f, M_\phi^*(k_z)\rangle_{H^2} = \langle \phi f, k_z\rangle_{H^2} = \phi(z)f(z) = \langle f, \overline{\phi(z)}k_z\rangle_{H^2}$$

for all $f \in H^2(\mathbb{D})$, so that $M_\phi^*(k_z) = \overline{\phi(z)}\,k_z$. Let $U := \{z \in \mathbb{D};\ |\phi(z)| < 1\}$ and $V := \{z \in \mathbb{D};\ |\phi(z)| > 1\}$. If ϕ is non-constant and $\phi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$, the open sets U and V are both non-empty by the open mapping theorem for analytic functions. In view of Corollary 1.10, it is enough to show that $\operatorname{span}\{k_z;\ z \in U\}$ and $\operatorname{span}\{k_z;\ z \in V\}$ are dense in $H^2(\mathbb{D})$. But this is clear, since if $f \in H^2(\mathbb{D})$ is orthogonal to k_z either for all $z \in U$ or for all $z \in V$ then f vanishes on a non-empty open set and hence is identically zero.

Conversely, assume that M_ϕ^* is hypercyclic (so that ϕ is certainly non-constant). Then $\|M_\phi\| = \|M_\phi^*\| > 1$, hence $\sup_{z\in\mathbb{D}} |\phi(z)| > 1$. Moreover, we also have $\inf_{z\in\mathbb{D}} |\phi(z)| < 1$. Indeed, if we assume that $\inf_{z\in\mathbb{D}} |\phi(z)| \geq 1$ then $1/\phi \in H^\infty$ and

$M^*_{1/\phi}$ is not hypercyclic since $\|M^*_{1/\phi}\| = \|M_{1/\phi}\| \leq 1$; and since $M^*_{1/\phi} = (M^*_\phi)^{-1}$, Corollary 1.3 shows that M^*_ϕ is not hypercyclic either. Thus, we get

$$\inf_{z\in\mathbb{D}} |\phi(z)| < 1 < \sup_{z\in\mathbb{D}} |\phi(z)|,$$

which yields $\phi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$ by a simple connectedness argument. □

We now turn to the "supercyclic" analogues of Theorems 1.2 and 1.6. This is essentially a matter of including a multiplicative factor, so the proofs will be rather sketchy.

THEOREM 1.12 *Let X be a separable F-space, and let $T \in \mathfrak{L}(X)$. The following are equivalent:*

(i) *T is supercyclic;*
(ii) *For each pair of non-empty open sets $(U, V) \subset X$, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{K}$ such that $\lambda T^n(U) \cap V \neq \varnothing$.*

In that case, $SC(T)$ is a dense G_δ subset of X.

PROOF As before, let $(V_j)_{j\in\mathbb{N}}$ be a countable basis of open sets for X. Then one can write $SC(T) = \bigcap_j \bigcup_{\lambda,n} (\lambda T^n)^{-1}(V_j)$, and the proof is completed exactly as that of Theorem 1.2. □

The following definition and the theorem below are due to H. N. Salas [216].

DEFINITION 1.13 *Let X be a Banach space, and let $T \in \mathfrak{L}(X)$. We say that T satisfies the* ***Supercyclicity Criterion*** *if there exist an increasing sequence of integers (n_k), two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and a sequence of maps $S_{n_k} : \mathcal{D}_2 \to X$ such that:*

(1) *$\|T^{n_k}(x)\|\, \|S_{n_k}(y)\| \to 0$ for any $x \in \mathcal{D}_1$ and any $y \in \mathcal{D}_2$;*
(2) *$T^{n_k} S_{n_k}(y) \to y$ for each $y \in \mathcal{D}_2$.*

THEOREM 1.14 *Let $T \in \mathfrak{L}(X)$, where X is a separable Banach space. Assume that T satisfies the Supercyclicity Criterion. Then T is supercyclic.*

PROOF Let U and V be two non-empty open subsets of X. Pick $x \in \mathcal{D}_1 \cap U$ and $y \in \mathcal{D}_2 \cap V$. It follows from part (1) of Definition 1.13 that we can find a sequence of non-zero scalars (λ_k) such that $\lambda_k T^{n_k}(x) \to 0$ and $\lambda_k^{-1} S_{n_k}(y) \to 0$. (Assume that $\alpha_k := \|T^{n_k}(x)\|$ and $\beta_k := \|S_{n_k}(y)\|$ are not both 0. If $\alpha_k\beta_k \neq 0$, put $\lambda_k := \beta_k^{1/2}\alpha_k^{-1/2}$. Otherwise, take $\lambda_k := 2^k\beta_k$ if $\alpha_k = 0$ and $\lambda_k := 2^{-k}\alpha_k^{-1}$ if $\beta_k = 0$). Then, for large enough k, the vector $z := x + \lambda_k^{-1} S_{n_k}(y)$ belongs to U and $\lambda_k T^{n_k}(z)$ belongs to V. By Theorem 1.12, this shows that T is supercyclic. □

We illustrate the Supercyclicity Criterion with the following important example.

EXAMPLE 1.15 Let $B_{\mathbf{w}}$ be a **weighted backward shift** on $\ell^2(\mathbb{N})$; $B_{\mathbf{w}}$ is the operator defined by $B_{\mathbf{w}}(e_0) = 0$ and $B_{\mathbf{w}}(e_n) = w_n e_{n-1}$ for $n \geq 1$, where $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$ and $\mathbf{w} = (w_n)_{n\geq 1}$ is a bounded sequence of positive numbers. Then $B_{\mathbf{w}}$ is supercyclic. In particular, if e.g. $w_n \to 0$ as $n \to \infty$ then $B_{\mathbf{w}}$ is a supercyclic operator which has no hypercyclic multiple.

PROOF Let $\mathcal{D}_1 = \mathcal{D}_2 := c_{00}(\mathbb{N})$ be the set of all finitely supported sequences. Let $S_{\mathbf{w}}$ be the linear map defined on $\mathcal{D}_2$ by $S_{\mathbf{w}}(e_n) = w_{n+1}^{-1} e_{n+1}$ and, for each $k \in \mathbb{N}$, set $S_k := S_{\mathbf{w}}^k$. Then, the Supercyclicity Criterion is satisfied with respect to $(n_k) := (k)$ because $\|B_{\mathbf{w}}^k(x)\| = 0$ for large enough k and $B_{\mathbf{w}}^k S_k = I$ on $\mathcal{D}_2$.

If $w_n \to 0$ then $\|(\lambda B_{\mathbf{w}})^n\| = |\lambda|^n \sup_{i\in\mathbb{N}}(w_{i+1}\cdots w_{i+n}) \to 0$ as $n \to \infty$, for each fixed $\lambda \in \mathbb{C}$. Hence, no multiple of $B_{\mathbf{w}}$ can be hypercyclic. □

1.1.1 The hypercyclic comparison principle

We conclude this section by introducing the following well-known concepts; they will be used several times in the book.

DEFINITION 1.16 *Let $T_0 : X_0 \to X_0$ and $T : X \to X$ be two continuous maps acting on topological spaces X_0 and X. The map T is said to be a **quasi-factor** of T_0 if there exists a continuous map with dense range $J : X_0 \to X$ such that the diagram*

$$\begin{tikzcd} X_0 \arrow[r, "T_0"] \arrow[d, "J"'] & X_0 \arrow[d, "J"'] \\ X \arrow[r, "T"] & X \end{tikzcd}$$

*commutes, i.e. $TJ = JT_0$. When this can be achieved with a homeomorphism $J : X_0 \to X$ (so that $T = JT_0J^{-1}$), we say that T_0 and T are **topologically conjugate**. Finally, when T_0 and T are linear operators and the factoring map (resp. the homeomorphism) J can be taken as linear, we say that T is a **linear** quasi-factor of T_0 (resp. that T_0 and T are **linearly conjugate**).*

The usefulness of these definitions comes from the following simple but important observation: *hypercyclicity is preserved by quasi-factors and supercyclicity as well as the Hypercyclicity Criterion are preserved by linear quasi-factors. Moreover, any factoring map J sends hypercyclic points to hypercyclic points.* This is indeed obvious since, with the above notation, we have $O(J(x_0), T) = J(O(x_0, T_0))$ for any $x_0 \in X_0$. In J. H. Shapiro's book [220], this observation is called the **hypercyclic comparison principle**. See Exercise 1.14 for a simple illustration.

A particular instance of the hypercyclic comparison principle is the following useful remark: *if $T \in \mathfrak{L}(X)$ is hypercyclic and if $J \in \mathfrak{L}(X)$ has dense range and commutes with T then $HC(T)$ is invariant under J.*

1.2 Some spectral properties

In this section, we show that hypercyclic and supercyclic operators have some noteworthy spectral properties. We start with the following simple observation. Here and