1 General relativity preliminaries

In this chapter we assemble some of the elements of Einstein's theory of general relativity that we will be working with in later chapters. We assume that the geometric objects and equations that we list, as well as their interpretation, are already very familiar to readers.[1] The discussion below should serve simply as a checklist of a few of the basics that we need to pack with us before embarking on our voyage into numerical spacetime.

Throughout this book we adopt the $(-+++)$ metric signature together with all the sign conventions of Misner *et al.* (1973). Following that book, but in this chapter only, we will display a tensor in spacetime by a symbol in boldface when emphasizing its coordinate-free character, or by its components when the tensor has been expanded in a particular set of basis tensors. However, unlike that book, we will use Latin indices $a, b, \ldots$ instead of Greek letters to denote the spacetime indices of the tensor components, with the values of the indices running from 0 to 3. This choice anticipates a switch we will make to *abstract index notation* in all subsequent chapters of this book. We will introduce this switch in Section 2.1. We adopt the usual Einstein convention of summing over repeated indices. Finally, here and throughout we will use geometrized units in which both the gravitational constant and the speed of light are assigned the values of one, $G = c = 1$.

1.1 Einstein's equations in 4-dimensional spacetime

Cast of characters

The metric tensor of 4-dimensional spacetime (i.e., the 4-metric) is denoted by g_{ab} and determines the invariant interval (distance) between two nearby events in spacetime according to

$$ds^2 = g_{ab} dx^a dx^b. \tag{1.1}$$

Here dx^a are the differences in the coordinates x^a that label events, or points, in spacetime. For a flat spacetime, g_{ab} becomes the Minkowski metric η_{ab}. In Cartesian coordinates with $x^0 = t$, $x^1 = x$, $x^2 = y$ and $x^3 = z$, the Minkowski metric components are

$$\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1), \tag{1.2}$$

representing a global inertial or Lorentz frame.

[1] They are treated in depth in introductory textbooks on general relativity, such as Misner *et al.* (1973), Weinberg (1972), Wald (1984) and Carroll (2004), to name a few.

In general, the components of the metric tensor are given by the scalar dot products between the four basis vectors $\mathbf{e}_a$ that span the vector space tangent to the spacetime manifold,[2]

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b. \tag{1.3}$$

In a coordinate basis, the basis vectors are tangent vectors to coordinate lines and may be written as $\mathbf{e}_a = \partial/\partial x^a \equiv \partial_a$. Clearly coordinate basis vectors commute. It is sometimes useful to set up orthonormal basis vectors at a point (an orthonormal tetrad) for which

$$\mathbf{e}_{\hat{a}} \cdot \mathbf{e}_{\hat{b}} = \eta_{\hat{a}\hat{b}}. \tag{1.4}$$

We denote an orthonormal tetrad by carets. In general, orthonormal basis vectors do not form a coordinate basis and do not commute. However, in flat spacetime it is always possible to transform to coordinates which are everywhere orthonormal or globally inertial, whereby the metric is given by equation (1.2) everywhere. For a general spacetime, this is not possible. But we can always choose any particular event in spacetime to be the origin of a local inertial coordinate frame, where $g_{ab} = \eta_{ab}$ at that point and where, in addition, the first derivatives of the metric tensor at that point vanish, i.e., $\partial_a g_{bc} = 0$. An observer in such a coordinate frame is called a local inertial or local Lorentz observer and can use a coordinate basis that forms a local orthonornal tetrad to make measurements as in special relativity. In fact, such an observer will find that all the (nongravitational) laws of physics in this frame are the same as in special relativity ("Principle of Equivalence").

For any set of basis vectors, a 4-vector $\mathbf{A}$ can be expanded in contravariant components,

$$\mathbf{A} = A^a \mathbf{e}_a. \tag{1.5}$$

The scalar product of two 4-vectors $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \cdot \mathbf{B} = (A^a \mathbf{e}_a) \cdot (B^b \mathbf{e}_b) = g_{ab} A^a B^b. \tag{1.6}$$

Now introduce a set of basis 1-forms $\tilde{\omega}^a$ dual to the basis vectors $\mathbf{e}_a$. An arbitrary 1-form $\tilde{\mathbf{B}}$ can be expanded in its covariant components according to

$$\tilde{\mathbf{B}} = B_a \tilde{\omega}^a. \tag{1.7}$$

The scalar product of two 1-forms $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ is

$$\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}} = (A_a \tilde{\omega}^a) \cdot (B_b \tilde{\omega}^b) = g^{ab} A_a B_b, \tag{1.8}$$

where $g^{ab} = \tilde{\omega}^a \cdot \tilde{\omega}^b$ is the inverse of g_{ab}. A basis of 1-forms dual to the basis $\mathbf{e}_a$ always satisfies

$$\tilde{\omega}^a \cdot \mathbf{e}_b = \delta^a{}_b. \tag{1.9}$$

[2] Recall that the subscript a in $\mathbf{e}_a$ denotes the ath basis vector, and not the a-component of a basis vector. In 4-dimensional spacetime, there are four independent basis vectors.

Accordingly, the scalar product of a vector with a 1-form does not involve the metric, but only a summation over an index:

$$\mathbf{A} \cdot \tilde{\mathbf{B}} = (A^a \mathbf{e}_a) \cdot (B_b \tilde{\omega}^b) = A^a \delta_a{}^b B_b = A^a B_a. \tag{1.10}$$

The vector $\mathbf{A}$ carries the same information as the corresponding 1-form $\tilde{\mathbf{A}}$, and we often will not make a distinction between them. Their components are related by

$$A_a = g_{ab} A^b, \tag{1.11}$$

or

$$A^a = g^{ab} A_b. \tag{1.12}$$

A coordinate basis of 1-forms may be written $\tilde{\omega}^a = \widetilde{\mathbf{d}x}^a$; geometrically, the basis form $\widetilde{\mathbf{d}x}^a$ may be thought of as surfaces of constant coordinate x^a. An orthonormal basis $\tilde{\omega}^{\hat{a}}$ is denoted by a caret and satisfies the relation

$$\tilde{\omega}^{\hat{a}} \cdot \tilde{\omega}^{\hat{b}} = \eta^{\hat{a}\hat{b}}. \tag{1.13}$$

A particularly useful one-form is $\widetilde{\mathbf{d}f}$, the gradient of an arbitrary scalar function f. In a coordinate basis, it may be expanded according to $\widetilde{\mathbf{d}f} = \partial_a f \widetilde{\mathbf{d}x}^a$, whereby its components are ordinary partial derivatives. The scalar product between an arbitrary vector $\mathbf{v}$ and the 1-form $\widetilde{\mathbf{d}f}$ gives the directional derivative of f along $\mathbf{v}$

$$\mathbf{v} \cdot \widetilde{\mathbf{d}f} = (v^a \mathbf{e}_a) \cdot (\partial_b f \widetilde{\mathbf{d}x}^b) = v^a \partial_a f. \tag{1.14}$$

A change of basis is always allowed, whereby $\mathbf{e}_{a'} = \mathbf{e}_b M^b{}_{a'}, \tilde{\omega}^{a'} = M^{a'}{}_b \tilde{\omega}^b$. Here $||M^b{}_{a'}||$ is an arbitrary, nonsingular matrix; its inverse is $||M^{a'}{}_b|| = ||M_a^{b'}||^{-1}$. Under such a change, components of vectors and 1-forms transform according to

$$A^{a'} = M^{a'}{}_b A^b, \quad B_{a'} = B_b M^b{}_{a'}. \tag{1.15}$$

When both of the bases are coordinate bases, then $M^b{}_{a'} = \partial_{a'} x^b$.

The generalization of the above concepts to tensors of arbitrary rank is straightforward. A 4-vector $\mathbf{A}$ and 1-form $\tilde{\mathbf{B}}$ are both tensors of rank 1. An arbitrary tensor can be expanded in its components, given a set of basis vectors and corresponding basis 1-forms. As an example, a mixed rank-2 tensor $\mathbf{T}$ can be expanded in components according to $\mathbf{T} = T^a{}_b \mathbf{e}_a \tilde{\omega}^b$. Here $\mathbf{e}_a \tilde{\omega}^b$ is a direct, or outer, tensor product. The componets of $\mathbf{T}$ transform according to

$$T^{a'}{}_{b'} = M^{a'}{}_c T^c{}_d M^d{}_{b'}. \tag{1.16}$$

The covariant derivative of an arbitrary tensor $\mathbf{T}$ is also a tensor and it measures the change of $\mathbf{T}$ with respect to parallel transport. For the above example of a mixed rank-2 tensor with components $T^a{}_b$, the covariant derivative is a tensor of rank 3 and its components are[3]

$$\nabla_c T^a{}_b = \partial_c T^a{}_b + {}^{(4)}\Gamma^a{}_{dc} T^d{}_b - {}^{(4)}\Gamma^d{}_{bc} T^a{}_d, \tag{1.17}$$

where the quantities ${}^{(4)}\Gamma^a{}_{bc}$ are connection coefficients or, in the special case of coordinate bases, Christoffel symbols, associated with the spacetime metric g_{ab}. The connection coefficients measure the change in the basis vectors and 1-forms with respect to parallel transport. In a coordinate basis they are related to partial derivatives of the metric by[4]

$${}^{(4)}\Gamma^a{}_{bc} = g^{ad}\,{}^{(4)}\Gamma_{dbc} = \frac{1}{2} g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}), \tag{1.18}$$

where the above relation defines ${}^{(4)}\Gamma_{dbc}$. In a local Lorentz frame the Christoffel symbols vanish. The covariant derivative of a scalar function f is the gradient 1-form; in components, $\nabla_a f = \partial_a f$. The corresponding vector $\nabla^a f$ is normal to the hypersurface $f =$ constant.

Curvature is the true measure of the gravitational field. The Riemann curvature tensor is given by

$${}^{(4)}R^a{}_{bcd} = \partial_c {}^{(4)}\Gamma^a{}_{bd} - \partial_d {}^{(4)}\Gamma^a{}_{bc} + {}^{(4)}\Gamma^a{}_{ec} {}^{(4)}\Gamma^e{}_{bd} - {}^{(4)}\Gamma^a{}_{ed} {}^{(4)}\Gamma^e{}_{bc} \tag{1.19}$$

in a coordinate basis.[5] Curvature vanishes if and only if the spacetime is flat. Second covariant derivatives of tensor fields do not commute in general and their difference is related to the Riemann tensor, e.g., for any vector v^a

$$\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c = v_d\,{}^{(4)}R^d{}_{cab}. \tag{1.20}$$

The Riemann tensor obeys a number of symmetries and identities, such as

$${}^{(4)}R_{abcd} = -{}^{(4)}R_{bacd}, \qquad {}^{(4)}R_{abcd} = -{}^{(4)}R_{abdc}, \qquad {}^{(4)}R_{abcd} = {}^{(4)}R_{cdab} \tag{1.21}$$

as well as the cyclic identity

$${}^{(4)}R_{abcd} + {}^{(4)}R_{adbc} + {}^{(4)}R_{acdb} = 0 \tag{1.22}$$

and the Bianchi identities

$$\nabla_e {}^{(4)}R_{abcd} + \nabla_d {}^{(4)}R_{abec} + \nabla_c {}^{(4)}R_{abde} = 0. \tag{1.23}$$

The symmetric Ricci tensor and Ricci scalar are formed from the Riemann tensor:

$${}^{(4)}R_{ab} = {}^{(4)}R^c{}_{acb} \tag{1.24}$$

$${}^{(4)}R = {}^{(4)}R^a{}_a. \tag{1.25}$$

[3] Sometimes the components of the covariant derivative of a tensor are written with a semicolon as $T^a{}_{b;c} \equiv \nabla_c T^a{}_b$.
[4] The expression for a noncoordinate basis involves additional commutation coefficient terms; see, e.g., Misner *et al.* (1973), equation (8.24b).
[5] See Misner *et al.* (1973), equation (11.3), for the components in a noncoordinate basis.

The Ricci tensor ${}^{(4)}R_{ab}$ is thus the trace of the Riemann tensor. The "trace-free part" is called the Weyl conformal tensor ${}^{(4)}C_{abcd}$ and, in four dimensions, is given by

$$
{}^{(4)}C_{abcd} = {}^{(4)}R_{abcd} - \frac{1}{2}(g_{ac}{}^{(4)}R_{bd} - g_{ad}{}^{(4)}R_{bc} - g_{bc}{}^{(4)}R_{ad} + g_{bd}{}^{(4)}R_{ac}) + \frac{1}{6}(g_{ac}g_{bd} - g_{ad}g_{bc})^{(4)}R. \tag{1.26}
$$

It is invariant under conformal transformations and vanishes if and only if the metric is conformally flat (i.e., can be transformed to Minkowski spacetime by a conformal transformation). For manifolds with dimensions ≤ 3, the Weyl tensor is identically zero and the Ricci tensor completely determines the Riemann tensor. In vacuum spacetimes, the Weyl tensor and the Riemann tensor are identical (by virtue of Einstein's equations (1.32) below).

Geodesics

Freely-falling test particles move along geodesic curves in spacetime. The tangent vector u^a of a geodesic curve is parallel propagated, $u^b\nabla_b u^a = 0$. If we introduce coordinates to construct the trajectories and set $u^a = dx^a/d\lambda$, then the geodesic equation becomes

$$
0 = u^b\nabla_b u^a = \frac{d^2x^a}{d\lambda^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda}, \tag{1.27}
$$

where λ is an affine parameter along the curve. For timelike particles with finite rest-mass, we can identify u^a with the particle 4-velocity and λ with proper time. In this case the quantity $a^a = u^b\nabla_b u^a$ is the 4-acceleration of the particle and is zero for geodesic motion. To accommodate null particles with zero rest-mass, we can always define an affine parameter by setting $p^a = dx^a/d\lambda$, where p^a is the particle 4-momentum. In terms of p^a the geodesic equation can be written as

$$
0 = p^b\nabla_b p^a = \frac{dp^a}{d\lambda} + \Gamma^a{}_{bc}p^b p^c = 0, \tag{1.28}
$$

and may be expressed exactly as in the right-hand side of equation (1.27) in a coordinate representation.

The function

$$
L = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b, \tag{1.29}
$$

where $\dot{x}^a \equiv dx^a/d\lambda$, provides a useful Lagrangian for geodesics. That is, the Euler–Lagrange equations derived from $L = L(x^a, \dot{x}^a)$ yield equations (1.27). The canonically conjugate momentum to the coordinate x^a is defined by

$$
p_a \equiv \frac{\partial L}{\partial \dot{x}^a}, \tag{1.30}
$$

and is just a covariant component of the 4-momentum of a particle. If the metric is independent of any coordinate x^a, then L is independent of the coordinate and p_a is a constant of the motion. In this case we say that $\mathbf{e}_a = \partial_a$ is a Killing vector of the spacetime, in which case the component $p_a = \mathbf{P} \cdot \mathbf{e}_a$ is conserved, where $\mathbf{P}$ is the particle 4-momentum vector.

The importance of Riemann curvature is reflected in the behavior of two nearby, freely-falling particles moving along two nearby geodesics with nearly equal affine parameters. If $u^a = dx^a/d\lambda$ is the tangent vector to one of the geodesics and n^a is the differential vector connecting the particles at equal values of affine parameter, then n^a satisfies the equation of geodesic deviation,

$$u^c \nabla_c (u^d \nabla_d n^a) = -{}^{(4)}R^a_{\ cbd} n^b u^c u^d. \tag{1.31}$$

The quantity on the left measures the relative acceleration of the two particles and it will be zero if and only if the tidal gravitational field, measured by Riemann curvature, is zero.

The Einstein field equations

In general relativity, the gravitational field is measured by the curvature of spacetime, and curvature is generated by the presence of matter, or, more properly, mass-energy. The energy, momentum and stress of matter are represented by the symmetric energy-momentum, or stress-energy, tensor T^{ab}. All nongravitational sources of energy and momentum in the Universe contribute to T^{ab} – all particles, fluids, fields, etc. For pure vacuum spacetimes we have $T^{ab} = 0$.

Einstein's field equations of general relativity relate the geometry of spacetime to the local matter content in the Universe according to

$$G_{ab} = 8\pi T_{ab}, \tag{1.32}$$

where G_{ab} is the symmetric Einstein tensor defined by

$$G_{ab} = {}^{(4)}R_{ab} - \frac{1}{2} g_{ab} {}^{(4)}R. \tag{1.33}$$

As a consequence of the Bianchi identities (1.23), the covariant divergence of G_{ab} vanishes, $\nabla_b G^{ab} = 0$, so equation (1.32) automatically guarantees that

$$\nabla_b T^{ab} = 0. \tag{1.34}$$

Equation (1.34) is the equation of motion governing the flow of energy and momentum for the matter. This equation is the statement that the total energy-momentum of the Universe is conserved. Solving equation (1.32) completely determines the spacetime metric, up to coordinate (gauge) transformations.

Astute readers will notice that a cosmological constant term has been omitted from equation (1.32). This omission has occurred in spite of cosmological evidence[6] that there exists such a term, as Einstein originally proposed, and that the actual field equations are in fact

$$G_{ab} + \Lambda g_{ab} = {}^{(4)}R_{ab} - \frac{1}{2} g_{ab} {}^{(4)}R. \tag{1.35}$$

However, the tiny magnitude inferred for the cosmological constant Λ makes this term completely unimportant for determining the dynamical behavior of relativistic stars, black holes, and most of the applications we treat in this book. Only when considering problems on cosmological scales, like the expansion of the Universe (which certainly affects the the propagation of electromagnetic and gravitational waves produced by local sources at large redshift), or the growth of primordial fluctuations and large-scale structure in the early Universe, is the presence of the Λ term important. For the applications we discuss in this book, and unless specifically stated otherwise, the cosmological constant will be taken to be zero and we will assume that our sources are immersed in an asymptotically flat vacuum spacetime.[7]

Gravitational radiation

Gravitational waves are ripples in the curvature of spacetime that propagate at the speed of light. Once the waves move away from their source in the near zone, their wavelengths are generally much smaller than the radius of curvature of the background spacetime through which they propagate. The waves usually can be described by linearized theory in this far zone region. Introducing Minkowski coordinates, one has

$$g_{ab} = \eta_{ab} + h_{ab}, \quad |h_{ab}| \ll 1, \tag{1.36}$$

where we assume Cartesian coordinates and, ignoring any quasistatic contributions to the perturbations h_{ab} from weak-field sources, consider only the wave contributions. Defining the trace-reversed wave perturbation $\bar{h}_{ab}$ according to

$$\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2} h^c{}_c \eta_{ab}, \tag{1.37}$$

the key equation governing the propagation of a linear wave in vacuum is

$$\Box \bar{h}_{ab} \equiv \nabla^c \nabla_c \bar{h}_{ab} = 0, \tag{1.38}$$

6 Measurements from the Wilkinson Microwave Anisotropy Probe (WMAP) combined with the Hubble Space Telescope yield a value for the cosmological constant of $\Lambda = 3.73 \times 10^{-56}$ cm^{-2}, corresponding to $\Omega_\Lambda \equiv \Lambda/(3H_0)^2 = 0.721 \pm 0.015$, where $H_0 = 70.1 \pm 1.3$ km/s/Mpc is Hubble's constant; Freedman *et al.* (2001); Spergel *et al.* (2007); Hinshaw *et al.* (2009).

7 It is also possible to restore the cosmological constant, or a slowly-varying term that mimics its effects, by incorporating an appropriate matter source term on the right hand side of equation (1.32). Such a "dark energy" contribution might arise from the stress-energy associated with the residual vacuum energy density (Zel'dovich 1967), or from an as yet unknown cosmic field, like a dynamical scalar field, sometimes referred to as "quintessence" (see, e.g., Peebles and Ratra 1988; Caldwell *et al.* 1998; see Chapter 5.4 for a discussion of dynamical scalar fields).

assuming it satisfies the Lorentz gauge condition

$$\nabla_b \bar{h}^{ab} = 0. \tag{1.39}$$

The Lorentz gauge condition does not yet define the gauge uniquely. Using the remaining gauge freedom we can introduce the transverse-traceless or "TT" gauge, defined by

$$h^{TT}_{a0} = 0, \quad h^{TT\,a}{}_{a} = 0, \tag{1.40}$$

which is particularly useful for describing gravitational waves. Gravitational waves are completely specified by two dimensionless amplitudes, h_+ and $h_\times$, representing the two possible polarization states of a gravitational wave. In terms of the polarization tensors e^+_{ab} and $e^\times_{ab}$ we may write a general gravitational wave as

$$h^{TT}_{jk} = h_+ e^+_{ij} + h_\times e^\times_{ij}, \tag{1.41}$$

where the letters $i, j, k, \ldots$ run over spatial indices only. For example, for a linear plane wave propagating in the z-direction, the amplitudes h_+ and $h_\times$ are functions of $t - z$ only and the only nonvanishing components of the polarization tensors are

$$e^+_{xx} = -e^+_{yy} = 1, \quad e^\times_{xy} = e^\times_{yx} = 1. \tag{1.42}$$

A passing gravitational wave drives the relative acceleration of two nearby test particles at a spatial separation ξ_i,

$$\ddot{\xi}_j = \frac{1}{2}\ddot{h}^{TT}_{jk}\xi^k. \tag{1.43}$$

According to equation (1.43), the wave amplitude measures the relative strain between the particles, $\delta\xi/\xi \sim h$. Equation (1.43) is the basis of most gravitational wave detectors.

Gravitational waves carry energy and momentum. The effective stress-energy tensor for gravitational waves is

$$T^{\rm GW}_{ab} = \frac{1}{32\pi}\left\langle \partial_a h^{TT}_{jk}\,\partial_b h^{TT}_{jk}\right\rangle, \tag{1.44}$$

where $\langle\ \rangle$ denotes an average over several wavelengths and where repeated indices are summed. The power generated in the form of gravitational waves by a weak-field, slow-motion ($v \ll 1$) source is given to leading order by the quadrupole formula,

$$L_{GW} = -\frac{dE}{dt} = \frac{1}{5}\left\langle \mathcal{I}^{(3)}_{ij}\mathcal{I}^{(3)}_{ij}\right\rangle, \tag{1.45}$$

where $\mathcal{I}$ is the "reduced quadrupole moment tensor" of the emitting source, given by

$$\mathcal{I}_{ij} \equiv \int \rho\left(x_i x_j - \frac{1}{3}\delta_{ij} r^2\right) d^3x. \tag{1.46}$$

Here $\langle\ \rangle$ denotes an average over several periods of the source, and $r = (x^2 + y^2 + z^2)^{1/2}$. The superscript (3) in the above formula indicates the third time derivative, E is the energy of the source, and, once again, repeated indices are summed. The angular momentum of the source is also being carried off by gravitational waves at a rate

$$\frac{dJ_i}{dt} = -\frac{2}{5}\epsilon_{ijk}\left\langle \mathcal{I}^{(2)}_{jm}\mathcal{I}^{(3)}_{km}\right\rangle . \tag{1.47}$$

Note, however, that no angular momentum is carried off if the source is axisymmetric, a result that is quite general. In the slow-velocity, weak-field approximation, the gravitational wave perturbation as measured by a distant observer is given by

$$h^{TT}_{jk}(t, x_j) = \frac{2}{r}\mathcal{I}^{TT(2)}_{jk}(t-r) . \tag{1.48}$$

Here the "TT" part of the reduced mass quadrupole moment is evaluated at retarded time $t' = t - r$ and is found from

$$\mathcal{I}^{TT}_{jk} \equiv P_{jl}P_{km}\mathcal{I}_{lm} - \frac{1}{2}P_{jk}(P_{lm}\mathcal{I}_{lm}), \tag{1.49}$$

where $P_{jk} \equiv \delta_{jk} - n_j n_k$ is the projection tensor that projects out the "TT" components and $n_j = x_j/r$ is a unit vector along the direction of propagation. In the same limit, one can add a radiation-reaction potential $\Phi^{\rm react}$, given by

$$\Phi^{\rm react} = \frac{1}{5}\mathcal{I}^{(5)}_{jk}x^j x^k, \tag{1.50}$$

to the Newtonian potential in the equations of motion of the source.[8] Such a radiation-reaction potential correctly drains the source of energy and angular momentum at just the rate at which gravitational waves carry off these quantities, but otherwise does not properly account for the post-Newtonian motion of the source.

A self-consistent treatment of gravitational waves that correctly describes their generation in a strong gravitational field to all orders, their evolution in the near-zone and their ultimate emergence and propagation in the far-zone, requires the full machinery of numerical relativity. The same machinery automatically accounts for the back-reaction of the radiation on the source. Forging such machinery is one of the goals of this book.

1.2 Black holes

A black hole is a region of spacetime that cannot communicate with the outside Universe. The boundary of this region is a 3-dimensional hypersurface in spacetime (a spatial 2-surface propagating in time) called the surface of the black hole or the *event horizon*. Nothing can escape from the interior of a black hole, not even light. Spacetime singularities inevitably form inside black holes. Provided the singularity is enclosed by the event horizon, it is "causally disconnected" from the exterior Universe and cannot influence it. Einstein's equations continue to describe the outside Universe, but they break down inside the black hole due to the singularity.

[8] Burke (1971).

The most general stationary black hole solution to Einstein's equations is the analytically known Kerr–Newman metric.[9] It is uniquely specified by just three paramters: the mass M, angular momentum J and the charge Q of the black hole. Special cases are the Kerr metric ($Q = 0$), the Reissner–Nordstrom metric ($J = 0$) and the Schwarzschild metric ($J = 0$, $Q = 0$).

Schwarzschild black holes

The Schwarzschild solution[10] for a vacuum spherical spacetime may be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1.51}$$

Written in this form, the radial coordinate r is called the *areal* radius since it is related to the area $\mathcal{A}$ of a spherical surface at r centered on the black hole according to the Euclidean expression $r = (\mathcal{A}/4\pi)^{1/2}$. The Schwarzschild solution holds in the vacuum region of any spherical spacetime, including a spacetime containing matter; it thus applies to the vacuum exterior of a static or collapsing star (Birkhoff's theorem). The mass of this spacetime, as measured by a distant static observer in the vacuum exterior, is M. When the vacuum extends down to $r = 2M$, the exterior spacetime corresponds to a vacuum black hole of mass M. The black hole event horizon is located at $r = 2M$ and is sometimes called the *Schwarzschild radius*. It is also referred to as the "static limit", because static observers cannot exist inside $r = 2M$, and the "surface of infinite redshift", because photons emitted by a static source just outside $r = 2M$ will have infinite wavelength when measured by a static observer at infinity.

Schwarzschild geometry admits the two Killing vectors, $\mathbf{e}_t = \partial_t$ and $\mathbf{e}_\phi = \partial_\phi$. Freely-falling test particles in Schwarzschild geometry thus conserve their energy $E = -p_t$ and orbital angular momentum $l = p_\phi$. Circular orbits of test particles exist down to $r = 3M$. The energy and angular momentum of a particle of rest-mass μ in circular orbit are given by

$$(E/\mu)^2 = \frac{(r - 2M)^2}{r(r - 3M)}, \tag{1.52}$$

$$(l/\mu)^2 = \frac{Mr^2}{r - 3M}. \tag{1.53}$$

The circular orbit at $r = 3M$ corresponds to a photon orbit ($E/\mu \to \infty$). Circular Schwarzschild orbits are stable if $r > 6M$, unstable if $r < 6M$.

The singularity in the metric at $r = 2M$ is a coordinate singularity, removable by coordinate transformation, while the singularity at $r = 0$ is a physical spacetime singularity. In fact, the curvature invariant

$$I \equiv {}^{(4)}R_{abcd}\,{}^{(4)}R^{abcd} = 48M^2/r^6 \tag{1.54}$$

[9] Kerr (1963); Newman *et al.* (1965)
[10] Schwarzschild (1916).