CAMBRIDGE TRACTS IN MATHEMATICS

178 Analysis in Positive Characteristic

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University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

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www.cambridge.org
Information on this title: www.cambridge.org/9780521509770

First published 2009

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-50977-0 Hardback

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Preface

One of the natural options in the development of mathematical analysis is to investigate analogs of classical objects in a new environment. To obtain the latter, it suffices to change the ground field. For a nontrivial analysis, we need a nondiscrete topological field. If we confine ourselves to locally compact fields (which can be extended subsequently via algebraic closures and completions), then the spectrum of possibilities is not very wide – $\mathbb{R}$, $\mathbb{C}$, the fields $\mathbb{Q}_p$ of p-adic numbers and their finite extensions, and non-Archimedean local fields of positive characteristic, that is the fields of formal Laurent series with coefficients from finite fields $\mathbb{F}_q$.

Over $\mathbb{R}$ and $\mathbb{C}$, the whole of classical mathematics is developed. The p-adic fields constitute the environment for p-adic analysis, a rapidly developing branch of contemporary mathematics embracing analogs of classical function theory, Fourier analysis, differential equations, theory of dynamical systems, probability, etc. Note that there are two kinds of p-adic analysis. One of them deals essentially with functions from $\mathbb{Q}_p$ to $\mathbb{R}$ or $\mathbb{C}$, like characters or probability densities, and its main results remain valid for local fields of positive characteristic (see [59]). The second treats functions with non-Archimedean arguments and values, like polynomials and power series, and here the extension to fields of positive characteristic is much more complicated.

To see the essence of these difficulties, it suffices to notice that some of the simplest standard notions of analysis do not make sense in the case of characteristic $p > 0$. For example, $n! = 0$ in a field of characteristic p, if $n \geq p$. Similarly, $\frac{d}{dt}(t^n) = 0$ if p divides n, that is the classical differential calculus cannot be used to investigate important classes of functions. For the same reason, the standard formulas do not work if one wants to define analogs of the most important special functions, beginning with the exponential function.

The first steps in developing analysis over local fields of positive characteristic were made in 1935 by Carlitz [22] who found appropriate positive characteristic counterparts for the factorial, exponential, logarithm, and classical orthogonal polynomials. Carlitz's constructions were highly unusual for that time, and their algebraic meaning was understood only about 40 years later, within the theory of Drinfeld modules and their generalizations.

After that they became important elements of a branch of number theory, function field arithmetic; see the books by Goss [45], Rosen [94], and Thakur [111].

Though the initial paper by Carlitz contained important insights into analysis (like the difference operator, very close to a later notion of the Carlitz derivative), the active development of analysis in positive characteristic began much later, after the paper [43] by Goss (with an exposition of Carlitz's work in more modern language) and the paper [109] by Thakur who introduced analogs of the hypergeometric function and the hypergeometric differential equation. That opened the way for a systematic development of the counterparts of the theory of Fourier series, the theory of differential equations, etc. The main achievements in these and related directions are summarized in this book.

In contrast to the books by Goss [45] and Thakur [111], whose material has some common features with the present one, this book is intended for specialists in analysis, from the graduate student level. Therefore only the most basic knowledge of algebra is presumed; in any case, an explanation of any algebraic notion, not given explicitly in the text, can be found in Lang's *Algebra*. In order to make the material more elementary, most notions are introduced on the local field level (though in some cases their global field interpretation is also outlined). The author's aim was to reach analysts who do not usually read books or papers in algebraic number theory and algebraic geometry. Accordingly, the author did not try to touch on some nonelementary material covered exhaustively in [45, 111], like the zeta and gamma functions and their arithmetic applications (a new analog of the zeta function, appearing in Chapter 4, is still only an example of an application of differential equations with the Carlitz derivatives). Some results requiring involved algebraic techniques are given without proofs.

Here is a short description of the contents. In Chapter 1, we introduce and study systems of functions on a local field of positive characteristic, which are crucial for solving various specific problems of analysis – the additive Carlitz polynomials and hyperdifferentiations. These functions are $\mathbb{F}_q$-linear; a method of lifting a $\mathbb{F}_q$-linear basis to a basis of a space of continuous functions was invented by Carlitz (1940) and found to be actually a very general construction ("the digit principle") by Conrad (2000). We describe both the general results and some specific cases. Then we consider the first special functions, the Carlitz exponential and logarithm, and the Carlitz module, and show how the above material appears in the positive characteristic analog of canonical commutation relations of quantum mechanics.

Chapter 2 deals mainly with the representation of functions by their expansions in a series of Carlitz polynomials, the correct understanding of the smoothness and analyticity of functions and their characterizations in terms of Fourier–Carlitz coefficients, the first notions of differential and integral calculus, which agree (in contrast to the classical ones) with the positive characteristic framework. The differential calculus is then extended to a kind of Rota's umbral calculus.

Given the notion of a derivative, it is natural to study differential equations, and that is done in Chapter 3, including analogs of Cauchy's existence and uniqueness theorems for analytic solutions, counterparts of regular singularity theory and partial differential equations of evolution type.

Some special functions satisfying differential equations with Carlitz derivatives are discussed in Chapter 4. Finally, in Chapter 5 we introduce and study rings of differential operators of the above kind. These rings can be seen as analogs of the Weyl algebras, though some of their properties are quite different – in fact, already the Carlitz derivative is, for instance, nonlinear (actually, $\mathbb{F}_q$-linear). Nevertheless, there exists a notion of quasi-holonomic modules over these rings, having some common features with holonomic modules in the sense of Bernstein, and connected to some special functions in the spirit of Zeilberger's theory.

The author is grateful to D. Goss, W. H. Schikhof and D. S. Thakur for their helpful consultations in non-Archimedean analysis and function field arithmetic.

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