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Some ideas from the theory of elasticity

1.1 Introduction

In this book we will use the linearized theory of elasticity to obtain solutions to some specialized problems in soil mechanics and foundation engineering. For beginning students in geotechnical engineering this may seem like a strange objective. One of the first things we learn about soil response is that, in general, it is neither linear nor elastic. The stress-strain behavior of soil is usually found to be hysteretic and highly nonlinear. It seems quite reasonable then to raise the question of how linear elasticity can be profitably used in relation to soils.

The answer to this question is two-fold. First, elasticity will be a convenient tool, and, in the busy and pressurized world of geotechnical engineering, convenient, reliable, and speedy results are a very positive advantage. The linear theory of elasticity possesses a long history, which has been distinguished not only by firm mathematical foundations but also by the solution of a large number of useful practical problems. Some of these solutions are particularly well-suited to the types of loadings and geometries we encounter in foundation engineering. To have a ready-made solution that either can be used immediately, or with only minor modification, is clearly a significant advantage. However, if the only advantages the theory of elasticity had to offer were convenience and speed, then it would remain an unused tool in the geotechnical engineer's tool kit. The second reason we use elastic theory is because most soils will behave approximately like a linear elastic material, provided the stresses they are subjected to are relatively *small*. By small, we mean the level of shearing stress within the soil deposit is considerably lower than its ultimate strength. This is often the case in many problems in foundation engineering. Usually foundations are designed with factors of safety of three or more. This suggests the stress level, in a general sense, is about one-third of the ultimate soil strength. If we consider a typical stress-strain curve for a fine-grained soil, such as illustrated in Figure 1.1, we might well con-

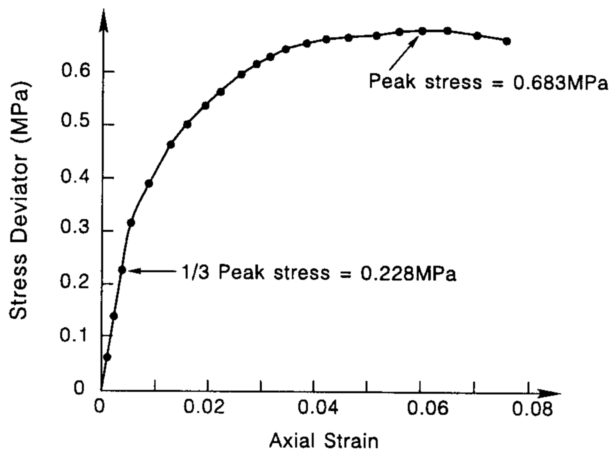

Figure 1.1 Stress deviator versus axial strain taken from results of an actual drained, stress-controlled triaxial test on a uniform beach sand. Note linearity of initial stress-strain response.

clude the behavior is at least approximately linear over the lower one-third of the peak stress range. Thus, for problems in which the soil is not subjected to stresses larger than about one-third the ultimate strength, elastic theory is not only convenient but also offers a rational approximation to the load-deformation behavior we may expect.

There are two other attractions in geotechnical engineering associated with elastic analysis. Elasticity solutions will often give insight into the "mechanics" of a problem which might not be available using other methods. An understanding of elasticity and its use in geotechnical problems frequently has led to new and innovative solutions to old problems. This is particularly true in certain fields such as in-situ testing of soils. The second attraction elasticity offers is that we may use elastic solutions as a check on more sophisticated computer-based solutions. Computational methods can offer an extremely wide range of problem solutions, but it is always comforting for the computer user to see that the program utilized gives correct answers to elastic problems for which the exact solution is known a priori.

The types of geotechnical problems where elasticity will be useful are mostly confined to foundations of structures. Some typical examples are illustrated in Figure 1.2. A great many structures are founded on reinforced concrete footings or pads buried at relatively shallow depths beneath the ground surface. For these types of structures, elastic theory is well-suited to yield estimates for both the stresses in the foundation soil and the displacements or

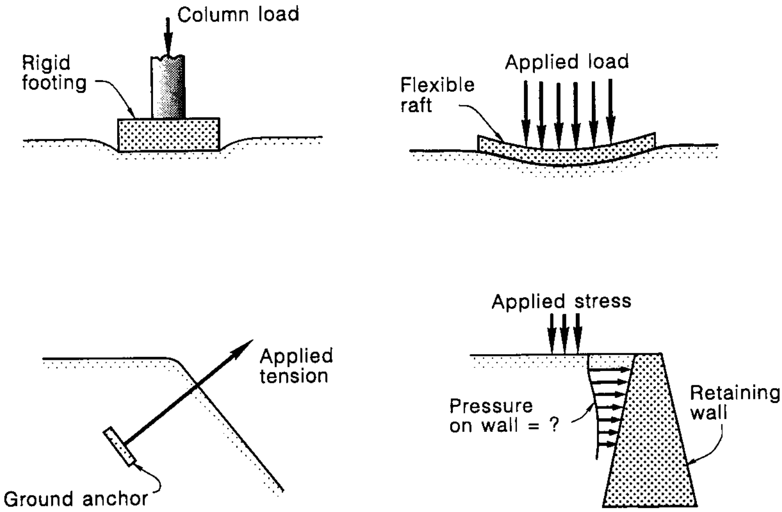

Figure 1.2 Some foundation engineering problems that can be fruitfully tackled using elastic analysis.

settlement of the structure itself. Elasticity is also a powerful tool in assessing the way in which the structure interacts with the soil. Since both the structure and the soil are deformable, but may have different stiffnesses, this problem, referred to as the soil-structure interaction problem, leads to many interesting results.

Practically all text books on foundation engineering contain some material based on elastic analysis. There are two books, however, which are of particular interest. The first is a book written by Harry Poulos and the late Ted Davis called *Elastic Solutions in Soil and Rock Mechanics*, published in 1974. This book is a compendium of solutions gathered from various sources and cataloged for easy reference. The second book is by Patrick Selvadurai, titled *Elastic Analysis of Soil-Foundation Interaction*, published in 1979. This is an advanced text giving a very complete description of the soil-structure interaction problem with formulations and solutions to many problems of practical interest. Both of these books are excellent sources for anyone interested in using linear elasticity in geotechnical engineering. However, both are different from the book you are now holding. This book is designed to introduce both undergraduate and graduate students to the subject. It is not exhaustive in content, and hopefully not exhausting to read. It will touch on a few interesting problems, but omit others. And it will give only a cursory description of the development of elasticity theory. Where we think a slightly more detailed exposition may be of interest to the more mathematically minded reader, it will be presented in Appendixes. In this text we primarily want to concentrate on how elasticity can be used rather than on why it came about. Hence, the title of this chapter, "Some Ideas From the Theory of Elasticity," means exactly what it says.

1.2 The notion of a continuum

The linearized theory of elasticity is a branch of a larger discipline called continuum mechanics. A very lucid definition of continuum mechanics was given by A. J. M. Spencer in his 1966 Inaugural Lecture as Professor of Theoretical Mechanics at the University of Nottingham.

> Continuum mechanics means mechanics of continuous media, and a continuous medium is a medium which occupies every point of a continuous region of space. In continuum mechanics we treat our materials – gases, liquids and solids – as though they completely fill up the regions they occupy, with no holes or gaps. This is not what materials are really like. They are made of molecules, and atoms, which in turn are made of smaller particles, and even these particles are not particles in the sense of being little hard lumps of matter. By far the greater part of any piece of any material is empty, and on the atomic scale the chance of any particular point being occupied by matter is small. Continuum mechanics pretends to ignore all this, and in effect smears the atoms, molecules and so on smoothly and uniformly over the region which we suppose our material to occupy. In doing this, of course, we are once again introducing a mathematical idealization; we are replacing a rather complicated idea of what materials are by a less accurate but much simpler one in much the same way as we make an idealization when we treat stars and planets as point masses when making astronomical calculations. There are two justifications for making this idealization. The first and most important is that it works. We can perform certain experiments on a material; on the basis of theory and these experiments we can make predictions about how the material will behave in other experiments....

Let us now examine the relevance of the notion of a continuum in the context of a soil. If we were to divide soil into blocks of one meter on a side we would intuitively expect a typical block to represent the constitution of the soil in some average sense. If we then continued to cut the block into smaller and smaller parts we would quickly find a point at which one part might consist entirely of a particular soil mineral such as quartz, while another part might consist entirely of water, or for that matter, another mineral such as feldspar. So technically a soil can never be a continuum, but this is not a serious obstacle. What we must do in order to treat soil as a continuum, or as Spencer remarked "to make it work," is to agree to apply our results only to volumes sufficiently large enough to encompass significant numbers of soil particles. This will clearly be the case for any reasonable problem in foundation engineering where the characteristic dimension of a foundation will be of the order of meters, whereas the characteristic dimension of a soil could range from 2 μm (e.g., a clay) to 50 mm (gravel).

One consequence of dealing with a continuum is that we can use the concept of a limit. For example, in a continuum the *mass density* ρ is defined as a limit

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V}$$

where ΔV is a volume and ΔM the mass of material contained in ΔV. In the limit process we shrink ΔV toward zero volume around a particular point. We will identify the point by its position in a three-dimensional Cartesian coordinate system, and we'll denote the position by $\boldsymbol{x}$. The components corresponding to $\boldsymbol{x}$ may be (x, y, z) in a rectangular coordinate system, or (r, θ, z) in a cylindrical coordinate system, or other coordinate names in systems of other flavors. Since the volume ΔV shrinks up to the point $\boldsymbol{x}$, we say $\rho = \rho(\boldsymbol{x})$. In more general problems, time may also be involved and $\rho = \rho(\boldsymbol{x},t)$. The only reservation we need in order to apply all of these ideas to soil is that, rather than take the limit as $\Delta V \to 0$, we will agree to consider finite volumes and restrain the limiting procedure to $\Delta V \to \Delta V_o$, where ΔV_o is sufficiently large to contain a significant number of soil particles.

The concept of a "permissible volume" ΔV_o becomes quite important especially when dealing with the mechanical testing of soils for the determination of their strength and deformability characteristics. For example, in the triaxial testing of soils such as clay, silt, and sand, the dimensions of a cylindrical sample of the soil can be 75 mm in diameter and 150 mm in length. Clearly, the same sample dimensions will not apply when testing granular materials such as gravel with particle sizes of the order of 50 mm. Here, the sample size must be substantially larger if we are to be assured that the results derived from the experiments conform to our notion of a continuum description.

1.3 Deformations of a continuum

When dealing with continua, one important aspect is the description of its deformations. We will need to be able to precisely describe the *deformations* that a continuum may experience as a result of action from outside forces. The term deformation refers both to the motion of a particular particle in the continuum and to the overall motion of the continuum itself. To be a little more specific, suppose our continuum is a *body*, B, which we could illustrate as a generic potato-ish shape in Figure 1.3. Later on we will be much more specific about the configuration of the body, but for the time being the gen-

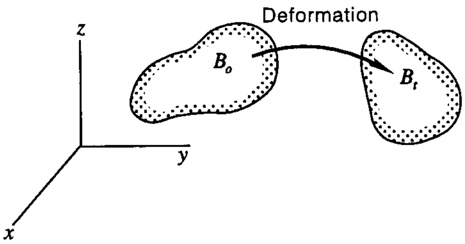

Figure 1.3 The reference configuration B_o and the deformed configuration B_t for a generic elastic body.

eral shape is all we need. We will let the body have a *reference configuration*, and this will be its shape and position when it is at rest, free from load. We will denote the reference configuration by B_o. We can also talk about its *deformed configuration*, after the loads have been applied, and we shall denote this by B_t. The subscript t here refers to time. The configuration B_t gives us the shape and position of the body at time t. We may need to be specific about t in cases where the applied loads vary with time. In Figure 1.3, the reference and deformed configurations are both shown and are linked by a generalized idea of deformation, shown as an arrow linking the two configurations.

How can we precisely describe the deformation? One way to accomplish this is to introduce a vector, called the *displacement vector* $\boldsymbol{u}$, which joins the position of a particular particle in the reference configuration to its position in the deformed configuration. A typical displacement vector is illustrated in Figure 1.4. If we define a displacement vector for every particle in the body, then the complete set of vectors forms a *vector field*, and we can write

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t) \tag{1.1a}$$

to show that $\boldsymbol{u}$ depends both on the position $\boldsymbol{x}$ and the time t.

It is also convenient to use an indicial notation to present the dependent and independent variable encountered in the presentation of basic results. For example, in indicial notation, the position vector is denoted by x_i where the subscript i can take the values 1, 2, 3, and for convenience we can denote $x_1 = x$, $x_2 = y$, $x_3 = z$. Consequently, the displacement vector $\boldsymbol{u}$ can be written as

$$u_i = u_i(x_j,t). \tag{1.1b}$$

In using the indicial notation we employ the summation convention adopted by Einstein in that if two indices are repeated in an equation, summation is

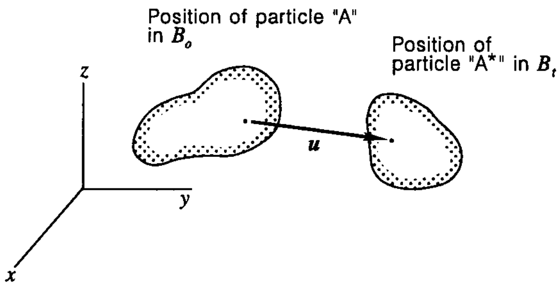

Figure 1.4 The displacement vector $\boldsymbol{u}$ joins the position "A" for a particle in the reference configuration to its position "A*" in the deformed configuration.

carried out over the repeated index unless otherwise explicitly stated to the contrary.

It is natural to wonder about the position $\boldsymbol{x}$ in eq. (1.1). Is $\boldsymbol{x}$ the position of the particle in the reference configuration, or is it the position taken by the particle in the deformed configuration? In a general development of continuum mechanics, $\boldsymbol{x}$ in eq. (1.1) would be the position in the reference configuration, but, in the linear theory of elasticity, we assume the *deformations are small*, so small that the distinction between the positions in the two configurations will not be important. That does not mean $\boldsymbol{u}$ itself is unimportant, only that whether we use the reference configuration or the deformed configuration positions as an independent variable is not important.

If $\boldsymbol{u}$ is known for a certain deformation, then we really have everything we need to know. The vector field $\boldsymbol{u}$ gives us the deformation of the body as a whole, and, if we are interested in a specific particle, then $\boldsymbol{u}(\boldsymbol{x},t)$ evaluated for the appropriate $\boldsymbol{x}$ gives us the displacement of the particle as a function of time. Later on, when we are dealing with foundations, $\boldsymbol{u}$ will tell us how much the foundation moves. The vertical component of $\boldsymbol{u}$, evaluated at a point immediately beneath the foundation, will give the *settlement* of that point.

1.4 Deformation and strain

We intuitively feel that deformations may lead to *strains* within a body. Before we can consider strains, however, we need to realize that some deformations will not cause strains. These are deformations called *rigid body deformations*, and they consist of either *rigid translations* or *rigid rotations*. A rigid translation is any deformation that does not depend on $\boldsymbol{x}$. Thus, if $\boldsymbol{u}$ for every $\boldsymbol{x}$ is the same, the body must be undergoing a rigid translation. Rigid rotations rotate the body about a fixed axis. We can be more specific about

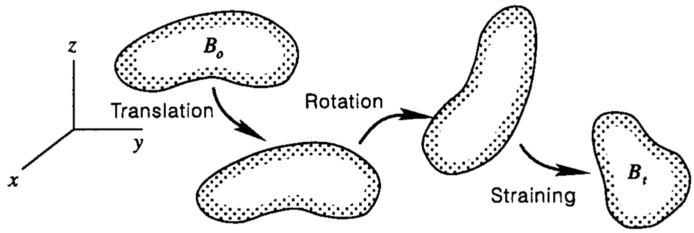

Figure 1.5 Any deformation can be decomposed into a sequence of rigid translation followed by rigid rotation followed by straining.

rotations in a moment. For the time being, the important thing to realize is that *any* deformation can be broken down into, at most, a rigid translation followed by a rigid rotation followed by straining. This idea is shown schematically in Figure 1.5.

The difference between straining and the two rigid motions is this: Strains result in changes in length or shape within the body; rigid motions do not. Suppose we consider two particles within the body that are quite close together in the reference configuration. Let the line that joins these particles be $d\boldsymbol{x}$. We can think of $d\boldsymbol{x}$ as a small filament of the material, and we can examine what happens to this filament during the deformation. If there is any rigid translation, the filament will move, but its orientation will remain unchanged and its length will not be changed either. If there is also rigid rotation, the filament will change its orientation, but its length will still be unchanged. If the filament is stretched or compressed in length, then the body is undergoing straining. We call the change in length divided by the original length the *extensional strain*.

$$\text{extensional strain} = \frac{\text{change in filament length}}{\text{original filament length}}$$

Changes in shape also result in strains. Consider two material filaments $d\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$ which, in the reference configuration, lie at right angles to each other, as in Figure 1.6. If the angle between the two filaments changes during the deformation, then, even if both filaments still have their original length, there has been straining. This is called *shear straining*. The shear strain is defined as the decrease in the angle between the two filaments.

The question we now need to answer is this: How can we go about separating the strains from the rigid motions? Suppose we were given a specified displacement field $\boldsymbol{u}(\boldsymbol{x},t)$. We are aware $\boldsymbol{u}$ may consist of a rigid translation,

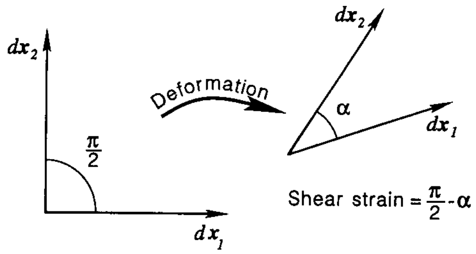

Figure 1.6 Shear strain is defined as the decrease in an initially right angle between two material filaments.

a rigid rotation, and straining. Only the strains will result in stresses within the body. If we want to characterize the stresses, we need to find the strains, and that means eliminating the rigid motions. The first step is to look at how $\boldsymbol{u}$ varies in the neighborhood of a point; i.e., we look at the partial derivatives of $\boldsymbol{u}$ rather than $\boldsymbol{u}$ itself. There will be nine partial derivatives in all. In a rectangular coordinate system each of the components u_x, u_y, and u_z will have three derivatives, one in each coordinate direction x, y, z. If the deformation were *only* a rigid translation, then $\boldsymbol{u}$ would be independent of the variables, and all the partial derivatives would be zero. So by considering the derivatives we have concentrated on rigid rotations and strains and eliminated rigid translations.

Next we need to separate rotations and strains. We can arrange the nine partial derivatives of $\boldsymbol{u}$ into a matrix called the *displacement gradient matrix*, $\nabla\boldsymbol{u}$.

$$\nabla\boldsymbol{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{1.2}$$

The matrix for $\nabla\boldsymbol{u}$ would look a little different if we were using a cylindrical or spherical coordinate system, but that need not worry us here. The next step is to decompose $\nabla\boldsymbol{u}$ into two matrices, one symmetric and one skew-symmetric. The symmetric matrix is called the *strain matrix*, $\boldsymbol{\epsilon}$, and is defined by

$$\boldsymbol{\epsilon} = \frac{1}{2}[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]. \tag{1.3}$$

Here the superposed T denotes the transpose of the matrix. The skew-symmetric matrix is called the *rotation matrix*, $\mathbf{\Omega}$, defined by

$$\mathbf{\Omega} = \frac{1}{2}[\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^T]. \tag{1.4}$$

Note that if we add $\boldsymbol{\epsilon}$ and $\mathbf{\Omega}$ we get $\nabla \boldsymbol{u}$. This decomposition into symmetric and skew-symmetric parts is unique and can be accomplished with any square matrix. Just as their names imply, $\boldsymbol{\epsilon}$ will correctly account for the strains in the body and $\mathbf{\Omega}$ will account for the rigid rotations.

We will write the components of the strain matrix like this

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \tag{1.5}$$

and bear in mind that symmetry of $\boldsymbol{\epsilon}$ implies $\epsilon_{xy} = \epsilon_{yx}$, $\epsilon_{xz} = \epsilon_{zx}$, and $\epsilon_{yz} = \epsilon_{zy}$. The diagonal components of $\boldsymbol{\epsilon}$, ϵ_{xx}, ϵ_{yy}, and ϵ_{zz}, are called the extensional strains. The off-diagonal components, ϵ_{xy}, etc., are called the shear strains. In terms of the partial derivatives of $\boldsymbol{u}$, we have

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{1.6}$$

$$\left.\begin{aligned} \epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \epsilon_{xz} = \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \epsilon_{yz} = \epsilon_{zy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \end{aligned}\right\} \tag{1.7}$$

We will see how these names come about by considering two simple examples.

The first example is shown in Figure 1.7. We have a material filament of length dx initially, aligned with the x-axis. (We can consider dx as just the scalar length of the filament whose vector description is $d\boldsymbol{x} = dx\,\hat{\boldsymbol{\imath}}$, where $\hat{\boldsymbol{\imath}}$ is the unit base vector in the x-direction.) This filament joins points A and B in the reference configuration. In the deformed configuration A has moved to A′ and B to B′. The filament length has now changed to $dx + (\partial u_x/\partial x)dx$,