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978-0-521-49789-3 - The Mathematics of Financial Derivatives: A Student Introduction

Paul Wilmott, Sam Howison and Jeff Dewynne

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Finance is one of the fastest growing areas in the modern banking and corporate world. This, together with the sophistication of modern financial products, provides a rapidly growing impetus for new mathematical models and modern mathematical methods; the area is an expanding source for novel and relevant 'real world' mathematics.

In this book the authors describe the modeling of financial derivative products from an applied mathematician's viewpoint, from modeling through analysis to elementary computation. A unified approach to modeling derivative products as partial differential equations is presented, using numerical solutions where appropriate. Some mathematics is assumed, but clear explanations are provided for material beyond elementary calculus, probability, and algebra.

This volume will become the standard introduction for advanced undergraduate students to this exciting new field.

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*For our children
Oscar, Zachary and Toby*

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The Mathematics of Financial Derivatives

A Student Introduction

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UNIVERSITY PRESS

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PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

<http://www.cambridge.org>

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First published 1995
Reprinted 1996 (twice with corrections)
Reprinted 1997 (twice) 1998, 1999, 2002

Printed in the United States of America

Typeset in TeX

A catalog record for this book is available from the British Library

Library of Congress Cataloging in Publication Data is available

ISBN 0 521 49789 2 paperback

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Frontmatter

[More information](#)

Contents

<i>Preface</i>	<i>page ix</i>
Part One: Basic Option Theory	1
1 An Introduction to Options and Markets	3
1.1 Introduction	3
1.2 What is an Option?	4
1.3 Reading the Financial Press	7
1.4 What are Options For?	11
1.5 Other Types of Option	13
1.6 Forward and Futures Contracts	14
1.7 Interest Rates and Present Value	15
2 Asset Price Random Walks	18
2.1 Introduction	18
2.2 A Simple Model for Asset Prices	19
2.3 Itô's Lemma	25
2.4 The Elimination of Randomness	30
3 The Black–Scholes Model	33
3.1 Introduction	33
3.2 Arbitrage	33
3.3 Option Values, Payoffs and Strategies	35
3.4 Put-call Parity	40
3.5 The Black–Scholes Analysis	41
3.6 The Black–Scholes Equation	44
3.7 Boundary and Final Conditions	46
3.8 The Black–Scholes Formulæ	48
3.9 Hedging in Practice	51
3.10 Implied Volatility	52

Cambridge University Press

978-0-521-49789-3 - The Mathematics of Financial Derivatives: A Student Introduction

Paul Wilmott, Sam Howison and Jeff Dewynne

Frontmatter

[More information](#)

vi		<i>Contents</i>
4	Partial Differential Equations	58
	4.1 Introduction	58
	4.2 The Diffusion Equation	59
	4.3 Initial and Boundary Conditions	66
	4.4 Forward versus Backward	68
5	The Black–Scholes Formulæ	71
	5.1 Introduction	71
	5.2 Similarity Solutions	71
	5.3 An Initial Value Problem	75
	5.4 The Formulæ Derived	76
	5.5 Binary Options	81
	5.6 Risk Neutrality	83
6	Variations on the Black–Scholes Model	90
	6.1 Introduction	90
	6.2 Options on Dividend-paying Assets	90
	6.3 Forward and Futures Contracts	98
	6.4 Options on Futures	100
	6.5 Time-dependent Parameters	101
7	American Options	106
	7.1 Introduction	106
	7.2 The Obstacle Problem	108
	7.3 American Options as Free Boundary Problems	109
	7.4 The American Put	110
	7.5 Other American Options	114
	7.6 Linear Complementarity Problems	115
	7.7 The American Call with Dividends	121
	Part Two: Numerical Methods	133
8	Finite-difference Methods	135
	8.1 Introduction	135
	8.2 Finite-difference Approximations	136
	8.3 The Finite-difference Mesh	138
	8.4 The Explicit Finite-difference Method	139
	8.5 Implicit Finite-difference Methods	144
	8.6 The Fully-implicit Method	144
	8.7 The Crank–Nicolson Method	155
9	Methods for American Options	165
	9.1 Introduction	165
	9.2 Finite-difference Formulation	167

Cambridge University Press

978-0-521-49789-3 - The Mathematics of Financial Derivatives: A Student Introduction

Paul Wilmott, Sam Howison and Jeff Dewynne

Frontmatter

[More information](#)

<i>Contents</i>	vii
9.3 The Constrained Matrix Problem	168
9.4 Projected SOR	169
9.5 The Time-stepping Algorithm	172
9.6 Numerical Examples	174
9.7 Convergence of the Method	176
10 Binomial Methods	180
10.1 Introduction	180
10.2 The Discrete Random Walk	183
10.3 Valuing the Option	187
10.4 European Options	187
10.5 American Options	189
10.6 Dividend Yields	191
Part Three: Further Option Theory	195
11 Exotic and Path-dependent Options	197
11.1 Introduction	197
11.2 Compound Options: Options on Options	199
11.3 Chooser Options	201
11.4 Barrier Options	201
11.5 Asian Options	202
11.6 Lookback Options	203
12 Barrier Options	206
12.1 Introduction	206
12.2 Knock-outs	207
12.3 Knock-ins	209
13 A Unifying Framework for Path-dependent Options	213
13.1 Introduction	213
13.2 Time Integrals of the Random Walk	214
13.3 Discrete Sampling	217
14 Asian Options	222
14.1 Introduction	222
14.2 Continuously Sampled Averages	223
14.3 Similarity Reductions	225
14.4 The Average Strike Option	226
14.5 Average Rate Options	230
14.6 Discretely Sampled Averages	233
15 Lookback Options	236
15.1 Introduction	236
15.2 Continuous Sampling of the Maximum	237

Cambridge University Press

978-0-521-49789-3 - The Mathematics of Financial Derivatives: A Student Introduction

Paul Wilmott, Sam Howison and Jeff Dewynne

Frontmatter

[More information](#)

viii	<i>Contents</i>
15.3 Discrete Sampling of the Maximum	243
15.4 Similarity Reductions	244
15.5 Some Numerical Examples	246
15.6 Two ‘Perpetual Options’	248
16 Options with Transaction Costs	252
16.1 Introduction	252
16.2 Discrete Hedging	252
16.3 Portfolios of Options	257
Part Four: Interest Rate Derivative Products	263
17 Interest Rate Derivatives	265
17.1 Introduction	265
17.2 Basics of Bond Pricing	265
17.3 The Yield Curve	268
17.4 Stochastic Interest Rates	270
17.5 The Bond Pricing Equation	270
17.6 Solutions of the Bond Pricing Equation	273
17.7 The Extended Vasicek Model of Hull & White	280
17.8 Bond Options	281
17.9 Other Interest Rate Products	282
18 Convertible Bonds	286
18.1 Introduction	286
18.2 Convertible Bonds	286
18.3 Convertible Bonds with Random Interest Rate	290
<i>Hints to Selected Exercises</i>	295
<i>Bibliography</i>	308
<i>Index</i>	312

Cambridge University Press

978-0-521-49789-3 - The Mathematics of Financial Derivatives: A Student Introduction

Paul Wilmott, Sam Howison and Jeff Dewynne

Frontmatter

[More information](#)

Preface

'Finance' is one of the fastest developing areas in the modern banking and corporate world. This, together with the sophistication of modern financial products, provides a rapidly growing impetus for new mathematical models and modern mathematical methods; the area is an expanding source for novel and relevant 'real-world' mathematics. The demand from financial institutions for well-qualified mathematicians is substantial, and there is a corresponding need for professional training of existing staff. Since 1992 the authors of this book have, in response, given graduate and undergraduate level courses on the subject. We have also organised a series of professional development courses for practitioners, held in Oxford and New York, with the assistance of Oxford University's Department for Continuing Education and the Oxford Centre for Industrial and Applied Mathematics. The material and notes from these courses became a book, *Option Pricing: Mathematical Models and Computation*, an advanced yet accessible account of applied and numerical techniques in the area of derivatives pricing.

Following the success of *Option Pricing* among financial practitioners, we have written this student-oriented version as an introduction to the subject. Our aim in *The Mathematics of Financial Derivatives: A Student Introduction* is to introduce the principles in a clear and readable way while leaving the more advanced topics and detailed practicalities, especially numerical issues, to the earlier book.

In what follows we describe the modelling of financial derivative products from an applied mathematician's viewpoint, from modelling through analysis to elementary computation. Some mathematics is assumed, but we explain everything that is not contained in the early calculus, probability and algebra courses of an undergraduate degree or equivalent in mathematics, physics, chemistry, engineering or similar subjects. We

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Paul Wilmott, Sam Howison and Jeff Dewynne

Frontmatter

[More information](#)

also give enough detail of the finance that the book can be read by mathematicians whose knowledge of financial markets is only sketchy. It is sufficiently self-contained that it could be used for a course on the subject, on its own or in conjunction either with a more probability-based text such as Duffie (1992) or with a more practically oriented book such as Hull (1993) or Gemmill (1992).

Our philosophy may be described briefly as follows:

- We present a unified approach to modelling many derivative products as partial differential equations. We make no more fuss over valuing an average strike option (a particularly exotic product) than over valuing the simplest option. There is a minimal use of fudges or approximations.
- We describe the theory of partial differential equations. We explain why they are one of the best approaches to modelling in many physical or financial subjects.
- We are happy to use numerical solutions. We would rather have an accurate numerical solution of the correct model than an explicit solution of the wrong model.

The authors of any book on financial models must decide at the outset what will be the mathematical basis of their approach. Essentially, this entails a decision on the amount of more or less rigorous analysis to incorporate, along with a choice whether or not to couch the discussion largely in terms of the language of stochastic processes. We feel that the interests of communication with our readers, especially those at the practical end of the subject, are best served by a relatively informal approach. We have therefore tried to stress the intuitive aspects of the subject, and this has led us naturally to emphasise the derivation and use of differential equations and associated numerical techniques. We hope that the rigour thereby forgone is compensated for by improved directness. We emphasise that there are excellent texts that fully cover more theoretical aspects of the subject.

The first chapter is an introduction to the subject of option pricing and markets. It is aimed at the reader who is new to the financial side of the subject and it contains no technical mathematical material.

Chapter 2 opens with a description of the movement of asset prices as a random walk. The underlying basis for the models used thereafter is that the future value of asset prices cannot be predicted with certainty. This does not mean that the movements cannot be modelled, only that any such model must incorporate a degree of randomness. We see that,

Cambridge University Press

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Frontmatter

[More information](#)*Preface*

xi

despite the random nature of asset prices, there are many problems for which we can make deterministic (that is, not probabilistic) statements. It is fortunate that such problems also happen to be the most interesting and important financially. Later in this chapter we informally describe the methods of stochastic calculus, building intuitively on the ideas of ordinary calculus.

In Chapter 3 the mathematical modelling becomes more explicitly related to option pricing. This chapter is perhaps the most important in the book. In it we present the cornerstone of the subject of option pricing: the derivation of the original Black–Scholes partial differential equation and boundary conditions for the value of an option.

The fact that partial differential equations prove to be central to our approach to financial modelling provides the motivation for the next four chapters. The type of partial differential equation that occurs most often in financial theory is the parabolic partial differential equation, the canonical example of which is the heat or diffusion equation. Posing the problems in the form of a parabolic partial differential equation means that we have nearly two centuries' worth of theory on which to call. Once the problem has been presented in such a form we may consider ourselves to be on well-known territory.

In Chapter 4 we discuss linear parabolic second order partial differential equations in quite general terms. In Chapter 5 the diffusion equation is dealt with in some detail. The general solution of the initial value problem on the whole line is derived and used to deduce the Black–Scholes formulæ for European call and put options. In Chapter 6 we discuss modifications necessary to account for the payment of dividends and then derive explicit formulæ for option prices when parameters are time-dependent. We also analyse futures and forward contracts, and options on them. The first part of the book concludes in Chapter 7 with a discussion of the free boundary problems that arise from models of American options where the possibility of early exercise gives rise to a free boundary. The theory of this chapter sets the scene for the numerical solution of free boundary problems.

Many of the models we derive do not admit closed form solutions. While it might be possible to modify them so that they can be solved in closed form, such modifications would probably have no basis in financial reality. We prefer, therefore, to accept that, for many practical problems, numerical methods of solution are necessary. The second part of the book is devoted to this topic.

In Chapter 8, we introduce finite differences for continuous models of

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Frontmatter

[More information](#)

European options. We demonstrate the explicit finite-difference method in some detail, and similarly describe the fully implicit and Crank–Nicolson methods. In Chapter 9 we discuss the numerical solution of free boundary problems for American options, again with particular emphasis on finite-difference methods. The final chapter on numerical methods, Chapter 10, gives details of discrete binomial models, which are a popular, albeit limited, alternative to finite differences.

Having dealt with the necessary basic theory, we come to some more advanced subjects in mathematical finance in the remainder of the book. Part 3 deals with so-called exotic and path-dependent options, and with the influence of transaction costs. We begin in Chapter 11 with an overview of exotic options, and we describe in more detail some quite simple contracts. Chapter 12 deals with another straightforward extension, this time to barrier options. In Chapter 13 we derive a general theory for options depending on history functionals of asset prices and give several examples in detail. Two path-dependent options have chapters to themselves: Asian options, which involve an average of the asset price, in Chapter 14, and lookback options, depending on the realised maximum or minimum, in Chapter 15. Finally, in Chapter 16 we consider a simple model for options in the presence of transaction costs. These can affect option prices quite significantly, and the model we discuss is of both practical importance and mathematical interest.

Throughout the first three parts of the book, the only random variable in our problems is the asset price. In the last part, we allow the interest rate to be unpredictable. Chapter 17 deals with the pricing of bonds and other interest rate derivatives; this entails the introduction of a simple stochastic model for the short-term interest rate. We conclude the book in Chapter 18 with a brief discussion of the valuation of convertible bonds; these bonds can have a value that depends on *two* random variables, an underlying asset and an interest rate.

Many people have offered us help, advice and encouragement during the production of both this book and *Option Pricing*. We are grateful in particular to the delegates on our courses, who gave us valuable insight into the workings of financial markets and made many helpful suggestions about the subjects of practical interest; to the graduate students and colleagues at the Oxford Centre for Industrial and Applied Mathematics; to the staff at the Oxford Centre for Continuing Professional Development; and lastly to the staff of Oxford Financial Press and Cambridge University Press for their continuing encouragement.

Cambridge University Press

978-0-521-49789-3 - The Mathematics of Financial Derivatives: A Student Introduction

Paul Wilmott, Sam Howison and Jeff Dewynne

Frontmatter

[More information](#)

Preface

xiii

The success of mathematics in finance depends heavily on the contributions of researchers in universities and financial institutions. It is intended that part of the royalties from the sales of this book will be used to fund a graduate scholarship for outstanding students who wish to study for a doctorate in the subject at the Oxford Centre for Industrial Applied Mathematics, Oxford University. For details of this award, which is intended as a supplement to the usual costs of fees and maintenance, write to: The Administrator, OCIAM, Mathematical Institute, 24–29 St. Giles', Oxford, OX1 3LB, UK.

Technical Point

Throughout the book, at the end of many sections and subsections, are scattered 'Technical Points'. As the name suggests, these items describe some of the more technical matters in our subject, matters which would disrupt the flow if contained in the main body of the text, yet which are too large to appear as footnotes. These items may be ignored on a first reading.

Occasionally, some words or phrases appear in **bold face**. This means that such a word or phrase is being defined. In some cases this definition is technical and in others simply descriptive.

Further Reading

There is a huge literature on financial mathematics. We have given selected references at the end of each chapter. Much of the material in the present book, especially in the chapters on numerical methods and exotic options, is covered in greater detail in *Option Pricing: Mathematical Methods and Computation*, also written by the present authors. This book is available from the publishers, Oxford Financial Press, PO Box 348, Oxford OX4 1DR, UK.

Exercises

Exercises are provided at the end of each chapter. Some of them are quite closely based on the text, while others, particularly in the later chapters, should be regarded as invitations to experiment. Hints to selected exercises are given at the end of the book.