Finance is one of the fastest growing areas in the modern banking and corporate world. This, together with the sophistication of modern financial products, provides a rapidly growing impetus for new mathematical models and modern mathematical methods; the area is an expanding source for novel and relevant 'real world' mathematics.

In this book the authors describe the modeling of financial derivative products from an applied mathematician's viewpoint, from modeling through analysis to elementary computation. A unified approach to modeling derivative products as partial differential equations is presented, using numerical solutions where appropriate. Some mathematics is assumed, but clear explanations are provided for material beyond elementary calculus, probability, and algebra.

This volume will become the standard introduction for advanced undergraduate students to this exciting new field.

The Mathematics of Financial Derivatives

For our children
Oscar, Zachary and Toby

The Mathematics of Financial Derivatives

A Student Introduction

PAUL WILMOTT
University of Oxford and Imperial College, London

SAM HOWISON
University of Oxford

JEFF DEWYNNE
University of Southampton

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Preface

'Finance' is one of the fastest developing areas in the modern banking and corporate world. This, together with the sophistication of modern financial products, provides a rapidly growing impetus for new mathematical models and modern mathematical methods; the area is an expanding source for novel and relevant 'real-world' mathematics. The demand from financial institutions for well-qualified mathematicians is substantial, and there is a corresponding need for professional training of existing staff. Since 1992 the authors of this book have, in response, given graduate and undergraduate level courses on the subject. We have also organised a series of professional development courses for practitioners, held in Oxford and New York, with the assistance of Oxford University's Department for Continuing Education and the Oxford Centre for Industrial and Applied Mathematics. The material and notes from these courses became a book, *Option Pricing: Mathematical Models and Computation*, an advanced yet accessible account of applied and numerical techniques in the area of derivatives pricing.

Following the success of *Option Pricing* among financial practitioners, we have written this student-oriented version as an introduction to the subject. Our aim in *The Mathematics of Financial Derivatives: A Student Introduction* is to introduce the principles in a clear and readable way while leaving the more advanced topics and detailed practicalities, especially numerical issues, to the earlier book.

In what follows we describe the modelling of financial derivative products from an applied mathematician's viewpoint, from modelling through analysis to elementary computation. Some mathematics is assumed, but we explain everything that is not contained in the early calculus, probability and algebra courses of an undergraduate degree or equivalent in mathematics, physics, chemistry, engineering or similar subjects. We also give enough detail of the finance that the book can be read by mathematicians whose knowledge of financial markets is only sketchy. It is sufficiently self-contained that it could be used for a course on the subject, on its own or in conjunction either with a more probability-based text such as Duffie (1992) or with a more practically oriented book such as Hull (1993) or Gemmill (1992).

Our philosophy may be described briefly as follows:

- We present a unified approach to modelling many derivative products as partial differential equations. We make no more fuss over valuing an average strike option (a particularly exotic product) than over valuing the simplest option. There is a minimal use of fudges or approximations.
- We describe the theory of partial differential equations. We explain why they are one of the best approaches to modelling in many physical or financial subjects.
- We are happy to use numerical solutions. We would rather have an accurate numerical solution of the correct model than an explicit solution of the wrong model.

The authors of any book on financial models must decide at the outset what will be the mathematical basis of their approach. Essentially, this entails a decision on the amount of more or less rigorous analysis to incorporate, along with a choice whether or not to couch the discussion largely in terms of the language of stochastic processes. We feel that the interests of communication with our readers, especially those at the practical end of the subject, are best served by a relatively informal approach. We have therefore tried to stress the intuitive aspects of the subject, and this has led us naturally to emphasise the derivation and use of differential equations and associated numerical techniques. We hope that the rigour thereby forgone is compensated for by improved directness. We emphasise that there are excellent texts that fully cover more theoretical aspects of the subject.

The first chapter is an introduction to the subject of option pricing and markets. It is aimed at the reader who is new to the financial side of the subject and it contains no technical mathematical material.

Chapter 2 opens with a description of the movement of asset prices as a random walk. The underlying basis for the models used thereafter is that the future value of asset prices cannot be predicted with certainty. This does not mean that the movements cannot be modelled, only that any such model must incorporate a degree of randomness. We see that, despite the random nature of asset prices, there are many problems for which we can make deterministic (that is, not probabilistic) statements. It is fortunate that such problems also happen to be the most interesting and important financially. Later in this chapter we informally describe the methods of stochastic calculus, building intuitively on the ideas of ordinary calculus.

In Chapter 3 the mathematical modelling becomes more explicitly related to option pricing. This chapter is perhaps the most important in the book. In it we present the cornerstone of the subject of option pricing: the derivation of the original Black–Scholes partial differential equation and boundary conditions for the value of an option.

The fact that partial differential equations prove to be central to our approach to financial modelling provides the motivation for the next four chapters. The type of partial differential equation that occurs most often in financial theory is the parabolic partial differential equation, the canonical example of which is the heat or diffusion equation. Posing the problems in the form of a parabolic partial differential equation means that we have nearly two centuries' worth of theory on which to call. Once the problem has been presented in such a form we may consider ourselves to be on well-known territory.

In Chapter 4 we discuss linear parabolic second order partial differential equations in quite general terms. In Chapter 5 the diffusion equation is dealt with in some detail. The general solution of the initial value problem on the whole line is derived and used to deduce the Black–Scholes formulæ for European call and put options. In Chapter 6 we discuss modifications necessary to account for the payment of dividends and then derive explicit formulæ for option prices when parameters are time-dependent. We also analyse futures and forward contracts, and options on them. The first part of the book concludes in Chapter 7 with a discussion of the free boundary problems that arise from models of American options where the possibility of early exercise gives rise to a free boundary. The theory of this chapter sets the scene for the numerical solution of free boundary problems.

Many of the models we derive do not admit closed form solutions. While it might be possible to modify them so that they can be solved in closed form, such modifications would probably have no basis in financial reality. We prefer, therefore, to accept that, for many practical problems, numerical methods of solution are necessary. The second part of the book is devoted to this topic.

In Chapter 8, we introduce finite differences for continuous models of

European options. We demonstrate the explicit finite-difference method in some detail, and similarly describe the fully implicit and Crank–Nicolson methods. In Chapter 9 we discuss the numerical solution of free boundary problems for American options, again with particular emphasis on finite-difference methods. The final chapter on numerical methods, Chapter 10, gives details of discrete binomial models, which are a popular, albeit limited, alternative to finite differences.

Having dealt with the necessary basic theory, we come to some more advanced subjects in mathematical finance in the remainder of the book. Part 3 deals with so-called exotic and path-dependent options, and with the influence of transaction costs. We begin in Chapter 11 with an overview of exotic options, and we describe in more detail some quite simple contracts. Chapter 12 deals with another straightforward extension, this time to barrier options. In Chapter 13 we derive a general theory for options depending on history functionals of asset prices and give several examples in detail. Two path-dependent options have chapters to themselves: Asian options, which involve an average of the asset price, in Chapter 14, and lookback options, depending on the realised maximum or minimum, in Chapter 15. Finally, in Chapter 16 we consider a simple model for options in the presence of transaction costs. These can affect option prices quite significantly, and the model we discuss is of both practical importance and mathematical interest.

Throughout the first three parts of the book, the only random variable in our problems is the asset price. In the last part, we allow the interest rate to be unpredictable. Chapter 17 deals with the pricing of bonds and other interest rate derivatives; this entails the introduction of a simple stochastic model for the short-term interest rate. We conclude the book in Chapter 18 with a brief discussion of the valuation of convertible bonds; these bonds can have a value that depends on *two* random variables, an underlying asset and an interest rate.

Many people have offered us help, advice and encouragement during the production of both this book and *Option Pricing*. We are grateful in particular to the delegates on our courses, who gave us valuable insight into the workings of financial markets and made many helpful suggestions about the subjects of practical interest; to the graduate students and colleagues at the Oxford Centre for Industrial and Applied Mathematics; to the staff at the Oxford Centre for Continuing Professional Development; and lastly to the staff of Oxford Financial Press and Cambridge University Press for their continuing encouragement.

The success of mathematics in finance depends heavily on the contributions of researchers in universities and financial institutions. It is intended that part of the royalties from the sales of this book will be used to fund a graduate scholarship for outstanding students who wish to study for a doctorate in the subject at the Oxford Centre for Industrial Applied Mathematics, Oxford University. For details of this award, which is intended as a supplement to the usual costs of fees and maintenance, write to: The Administrator, OCIAM, Mathematical Institute, 24–29 St. Giles', Oxford, OX1 3LB, UK.

Technical Point
Throughout the book, at the end of many sections and subsections, are scattered 'Technical Points'. As the name suggests, these items describe some of the more technical matters in our subject, matters which would disrupt the flow if contained in the main body of the text, yet which are too large to appear as footnotes. These items may be ignored on a first reading.

Occasionally, some words or phrases appear in **bold face**. This means that such a word or phrase is being defined. In some cases this definition is technical and in others simply descriptive.

Further Reading

There is a huge literature on financial mathematics. We have given selected references at the end of each chapter. Much of the material in the present book, especially in the chapters on numerical methods and exotic options, is covered in greater detail in *Option Pricing: Mathematical Methods and Computation*, also written by the present authors. This book is available from the publishers, Oxford Financial Press, PO Box 348, Oxford OX4 1DR, UK.

Exercises

Exercises are provided at the end of each chapter. Some of them are quite closely based on the text, while others, particularly in the later chapters, should be regarded as invitations to experiment. Hints to selected exercises are given at the end of the book.