Symmetry Methods for Differential Equations

Symmetry is the key to solving differential equations. There are many well-known techniques for obtaining exact solutions, but most of them are merely special cases of a few powerful symmetry methods. These methods can be applied to differential equations of an unfamiliar type; they do not rely on special "tricks." Instead, a given differential equation can be made to reveal its symmetries, which are then used to construct exact solutions.

This book is a straightforward introduction to symmetry methods; it is aimed at applied mathematicians, physicists, and engineers. The presentation is informal, with many worked examples. It is written at a level suitable for post-graduates and advanced undergraduates. The reader should be able to master the main techniques quickly and easily.

This text contains several new methods that will interest those whose research involves symmetries. In particular, methods for obtaining discrete symmetries and first integrals are described.

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To
Alison
Christopher
Rachel
and
Katy
who waited patiently for me
to come out of my study.
The wait is over.

Contents

Preface

There are many ingenious techniques for obtaining exact solutions of differential equations, but most work only for a very limited class of problems. How can one solve differential equations of an unfamiliar type?

Surprisingly, most well-known techniques have a common feature: they exploit *symmetries* of differential equations. It is often quite easy to find symmetries of a given differential equation (even an unfamiliar one) and to use them systematically to obtain exact solutions. Symmetries can also be used to simplify problems and to understand bifurcations of nonlinear systems.

More than a century ago, the Norwegian mathematician Sophus Lie put forward many of the fundamental ideas behind symmetry methods. Most of these ideas are essentially simple, but are so far reaching that they are still the basis of much research. As an applied mathematician, I have found symmetry methods to be invaluable. They are fairly easy to master and provide the user with a powerful range of tools for studying new equations. I believe that no one who works with differential equations can afford to be ignorant of these methods.

This book introduces applied mathematicians, engineers, and physicists to the most useful symmetry methods. It is aimed primarily at postgraduates and those involved in research, but there is sufficient elementary material for a one-semester undergraduate course. (Over the past five years, I have taught these methods to both undergraduates and postgraduates.) Bearing in mind the interests and needs of the intended readership, the book focuses on techniques. These are described and justified informally, without a "theorem–proof" format. I have tried to present the theory straightforwardly, sacrificing rigour and generality (where necessary) in order to communicate the most useful results clearly.

The topics are arranged so as to provide a graded introduction. Thus the reader can see symmetry methods applied at an early stage, without first having to absorb much new notation. As the book progresses, the methods are

generalized and extended. Practice is essential to develop skill in using symmetry methods; readers are urged to try the exercises at the end of each chapter. Solutions and hints for some exercises are available at the end of the book.

Here are some suggestions for those wishing to use this book as the basis of a lecture course. The first six chapters consist of core material on ordinary differential equations. In my experience, this is sufficient for a one-semester undergraduate course. For a postgraduate course, Chapters 8 and 9 (which deal with basic symmetry methods for partial differential equations) should also be included. I strongly recommend that students learn how to use an appropriate computer algebra package, because symmetry calculations can be lengthy (particularly for partial differential equations). I have briefly outlined some packages that are currently available at no cost to the user.

The remaining chapters outline some recent developments. These are selected on the grounds that they are widely applicable and easy to master. Some of these topics have not previously been described at an elementary level. I have omitted several techniques on the grounds that they are difficult to describe accurately without using complicated mathematical ideas. My aim throughout has been to enable the reader to become proficient in the most useful symmetry methods.

Peter E. Hydon
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