Symmetry Methods for Differential Equations
A Beginner's Guide

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Introduction to Symmetries

I know it when I see it.

(Justice Potter Stewart: Jacoblellis v. Ohio, 378 U.S. 184, 197 [1964])

1.1 Symmetries of Planar Objects

In order to understand symmetries of differential equations, it is helpful to consider symmetries of simpler objects. Roughly speaking, a symmetry of a geometrical object is a transformation whose action leaves the object apparently unchanged. For instance, consider the result of rotating an equilateral triangle anticlockwise about its centre. After a rotation of $2\pi/3$, the triangle looks the same as it did before the rotation, so this transformation is a symmetry. Rotations of $4\pi/3$ and 2π are also symmetries of the equilateral triangle. In fact, rotating by 2π is equivalent to doing nothing, because each point is mapped to itself. The transformation mapping each point to itself is a symmetry of any geometrical object: it is called the *trivial symmetry*.

Symmetries are commonly used to classify geometrical objects. Suppose that the three triangles illustrated in Fig. 1.1 are made from some rigid material, with indistinguishable sides. The symmetries of these triangles are readily found by experiment. The equilateral triangle has the trivial symmetry, the rotations described above, and flips about the three axes marked in Fig. 1.1(a). These flips are equivalent to reflections in the axes. So an equilateral triangle has six distinct symmetries. The isoceles triangle in Fig. 1.1(b) has two: a flip (as shown) and the trivial symmetry. Finally, the triangle with three unequal sides in Fig. 1.1(c) has only the trivial symmetry.

There are certain constraints on symmetries of geometrical objects. Each symmetry has a unique inverse, which is itself a symmetry. The combined action of the symmetry and its inverse upon the object (in either order) leaves

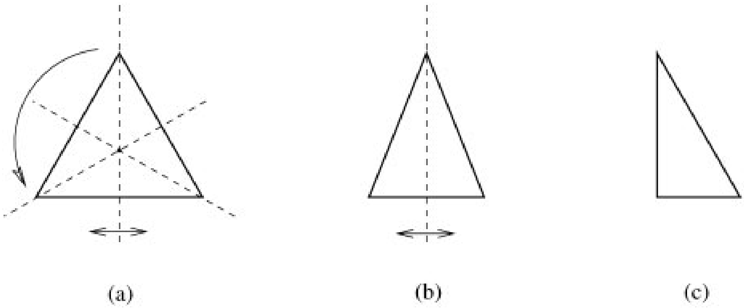

Fig. 1.1. Some triangles and their symmetries.

the object unchanged. For example, let Γ denote a rotation of the equilateral triangle by $2\pi/3$. Then Γ^{-1} (the inverse of Γ) is a rotation by $4\pi/3$.

For simplicity, we restrict attention to symmetries that are *smooth*. (This somewhat technical requirement is not greatly restrictive, and it frees us from the need to consider pathological examples.) If x denotes the position of a general point of the object, and if

$$\Gamma : x \mapsto \hat{x}(x)$$

is any symmetry, then we assume that $\hat{x}$ is infinitely differentiable with respect to x. Moreover, since Γ^{-1} is also a symmetry, x is infinitely differentiable with respect to $\hat{x}$. Thus Γ is a (C^∞) *diffeomorphism*, that is, a smooth invertible mapping whose inverse is also smooth.

Symmetries are also required to be *structure preserving*. It is usual for geometrical objects to have some structure which (loosely speaking) describes what the object is made from. To use an analogy from continuum mechanics, the structure is the constitutive relation for the object. Earlier, we considered symmetries of triangles made from a rigid material. The only transformations under which a triangle remains rigid are those which preserve the distance between any two points on the triangle, namely translations, rotations, and reflections (flips). These transformations are the only possible symmetries, because all other transformations fail to preserve the rigid structure. However, if the triangles are made from an elastic material such as rubber, the class of structure-preserving transformations is larger, and new symmetries may be found. For example, a triangle with three unequal sides can be stretched into an equilateral triangle, then rotated by $2\pi/3$ about its centre, and finally stretched so as to appear to have its original shape. This transformation is not a symmetry of a rigid triangle. Clearly, the structure associated with a geometrical object has a considerable influence upon the set of symmetries of the object.

In summary, a transformation is a symmetry if it satisfies the following:

(S1) The transformation preserves the structure.
(S2) The transformation is a diffeomorphism.
(S3) The transformation maps the object to itself [e.g., a planar object in the (x, y) plane and its image in the $(\hat{x}, \hat{y})$ plane are indistinguishable].

Henceforth, we restrict attention to transformations satisfying (S1) and (S2). Such transformations are symmetries if they also satisfy (S3), which is called the *symmetry condition*.

A rigid triangle has a finite set of symmetries. Many objects have an infinite set of symmetries. For example, the (rigid) unit circle

$$x^2 + y^2 = 1$$

has a symmetry

$$\Gamma_\varepsilon : (x, y) \mapsto (\hat{x}, \hat{y}) = (x \cos\varepsilon - y \sin\varepsilon,\ x \sin\varepsilon + y \cos\varepsilon)$$

for each $\varepsilon \in (-\pi, \pi]$. In terms of polar coordinates,

$$\Gamma_\varepsilon : (\cos\theta, \sin\theta) \mapsto \big(\cos(\theta + \varepsilon), \sin(\theta + \varepsilon)\big),$$

as shown in Fig. 1.2. Hence the transformation is a rotation by ε about the centre of the circle. It preserves the structure (rotations are rigid), and it is smooth and invertible (the inverse of a rotation by ε is a rotation by $-\varepsilon$). To prove that the symmetry condition (S3) is satisfied, note that

$$\hat{x}^2 + \hat{y}^2 = x^2 + y^2,$$

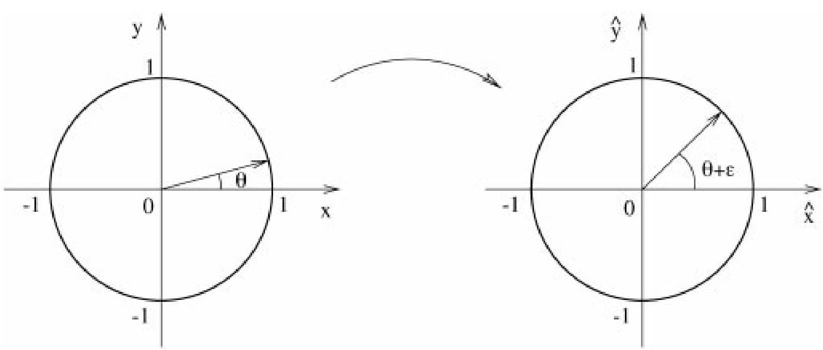

Fig. 1.2. Rotation of the unit circle.

and therefore

$$\hat{x}^2 + \hat{y}^2 = 1 \qquad \text{when} \quad x^2 + y^2 = 1.$$

The unit circle has other symmetries, namely reflections in each straight line passing through the centre. It is not difficult to show that every reflection is equivalent to the reflection

$$\Gamma_R : (x, y) \mapsto (-x, y)$$

followed by a rotation Γ_ε.

The infinite set of symmetries Γ_ε is an example of a *one-parameter Lie group*. This class of symmetries is immensely useful and is the key to constructing exact solutions of many differential equations. Suppose that an object occupying a subset of $\mathbb{R}^N$ has an infinite set of symmetries

$$\Gamma_\varepsilon : x^s \mapsto \hat{x}^s(x^1, \ldots, x^N; \varepsilon), \qquad s = 1, \ldots, N,$$

where ε is a real parameter, and that the following conditions are satisfied.

(L1) Γ_0 is the trivial symmetry, so that $\hat{x}^s = x^s$ when $\varepsilon = 0$.
(L2) Γ_ε is a symmetry for every ε in some neighbourhood of zero.
(L3) $\Gamma_\delta \Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}$ for every δ, ε sufficiently close to zero.
(L4) Each $\hat{x}^s$ may be represented as a Taylor series in ε (in some neighbourhood of $\varepsilon = 0$), and therefore

$$\hat{x}^s(x^1, \ldots, x^N; \varepsilon) = x^s + \varepsilon \xi^s(x^1, \ldots, x^N) + O(\varepsilon^2), \qquad s = 1, \ldots, N.$$

Then the set of symmetries Γ_ε is a one-parameter local Lie group. The term "local" (which we shall usually omit hereafter) refers to the fact that the conditions need only apply in some neighbourhood of $\varepsilon = 0$. Furthermore, the maximum size of the neighbourhood may depend on $x^s,\ s = 1, \ldots, N$. The term "group" is used because the symmetries Γ_ε satisfy the axioms of a group, at least for ε sufficiently close to zero. In particular, (L3) implies that $\Gamma_\varepsilon^{-1} = \Gamma_{-\varepsilon}$. Conditions (L1) to (L4) are slightly more restrictive than is necessary, but they allow us to start solving differential equations without becoming entangled in complexities.

Symmetries belonging to a one-parameter Lie group depend continuously on the parameter. As we have seen, an object may also have symmetries that belong to a discrete group. These *discrete symmetries* cannot be represented by a continuous parameter. For example, the set of symmetries of the equilateral triangle has the structure of the dihedral group D_3, whereas the two symmetries of the isoceles triangle form the cyclic group $\mathbb{Z}_2$. Discrete symmetries are useful in many ways, as described at the end of the book. Until then, we shall focus on parametrized Lie groups of symmetries, which are easier to find and use. For brevity, we refer to such symmetries as *Lie symmetries*.

For most of the time, we shall study the functions $\hat{x}^s$ directly, without reference to any ideas from group theory. Therefore it is convenient to simplify the notation by abbreviating

$$\Gamma_\varepsilon : (x^1, \ldots, x^N) \mapsto (\hat{x}^1, \ldots, \hat{x}^N) = \cdots$$

to

$$(\hat{x}^1, \ldots, \hat{x}^N) = \cdots.$$

Suffix notation is useful for stating general results, but we shall avoid using it in examples, as far as possible. Variables will be named $x, y, \ldots$ in preference to $x^1, x^2, \ldots$.

1.2 Symmetries of the Simplest ODE

What are the symmetries of ordinary differential equations (ODEs)? To begin to answer this question, consider the simplest ODE of all, namely

$$\frac{dy}{dx} = 0. \tag{1.1}$$

The set of all solutions of the ODE is the set of lines

$$y(x) = c, \qquad c \in R,$$

which fills the (x, y) plane. The ODE (1.1) is represented geometrically by the set of all solutions, and so any symmetry of the ODE must necessarily map the solution set to itself. More formally, the symmetry condition (S3) requires that the set of solution curves in the (x, y) plane must be indistinguishable from its image in the $(\hat{x}, \hat{y})$ plane, and therefore

$$\frac{d\hat{y}}{d\hat{x}} = 0 \qquad \text{when} \quad \frac{dy}{dx} = 0. \tag{1.2}$$

A smooth transformation of the plane is invertible if its Jacobian is nonzero, so we impose the further condition

$$\hat{x}_x \hat{y}_y - \hat{x}_y \hat{y}_x \neq 0. \tag{1.3}$$

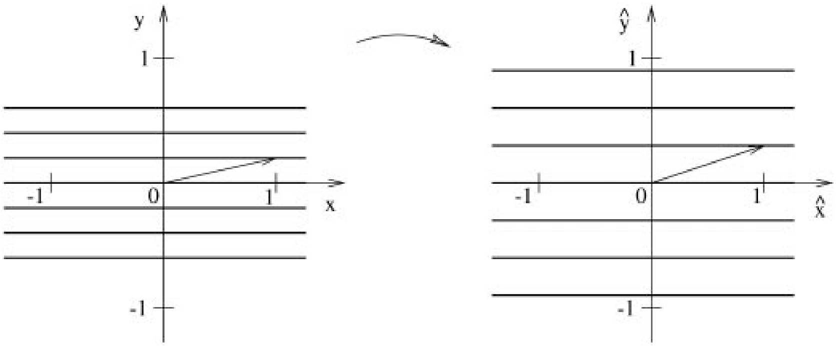

Fig. 1.3. Solutions of (1.1) transformed by a scaling, (1.5).

(Throughout the book, variable subscripts denote partial derivatives, e.g., $\hat{x}_x$ denotes $\frac{\partial \hat{x}}{\partial x}$.) A particular solution curve will be mapped to a (possibly different) solution curve, and so

$$\hat{y}(x, c) = \hat{c}(c), \qquad \forall\, c \in \mathbb{R}. \tag{1.4}$$

Here x is regarded as a function of $\hat{x}$ and c that is obtained by inverting

$$\hat{x} = \hat{x}(x, c).$$

The ODE (1.1) has many symmetries, some of which are obvious from Fig. 1.3. There are discrete symmetries, such as reflections in the x and y axes. Lie symmetries include scalings of the form

$$(\hat{x}, \hat{y}) = (x, e^{\varepsilon} y), \qquad \varepsilon \in \mathbb{R}. \tag{1.5}$$

[Figure 1.3 depicts the effect of the scalings (1.5) on only a few solution curves; if all solution curves could be shown, the two halves of the figure would be identical.] Every translation,

$$(\hat{x}, \hat{y}) = (x + \varepsilon_1, y + \varepsilon_2), \qquad \varepsilon_1,\ \varepsilon_2 \in \mathbb{R}, \tag{1.6}$$

is a symmetry. The set of all translations depends upon two parameters, ε_1 and ε_2. By setting ε_1 to zero, we obtain the one-parameter Lie group of translations in the y direction. Similarly, the one-parameter Lie group of translations in the x direction is obtained by setting ε_2 to zero. The set of translations (1.6) is a *two*-parameter Lie group, which can be regarded as a composition of the one-parameter Lie groups of translations parametrized by ε_1 and ε_2 respectively. Roughly speaking, symmetries belonging to an R-parameter Lie group can be regarded as a composition of symmetries from R one-parameter Lie groups.

Not every one-parameter Lie group is useful. For example, a translation (1.6) maps a solution curve $y = c$ to the curve $\hat{y} = c + \varepsilon_2$. If $\varepsilon_2 = 0$, any solution curve is mapped to itself by the symmetry. This is obvious, because translations in the x direction move points along the curves of constant y. Symmetries that map every solution curve to itself are described as *trivial*, even if they move points along the curves.

The ODE (1.1) is extremely simple, and so all of its symmetries can be found. Differentiating (1.4) with respect to x, we obtain

$$\hat{y}_x(x, c) = 0, \qquad \forall\, c \in \mathbb{R}.$$

Therefore, taking (1.3) into account, the symmetries of (1.1) are of the form

$$(\hat{x}, \hat{y}) = (f(x, y), g(y)), \qquad f_x \neq 0, \quad g_y \neq 0, \tag{1.7}$$

where f and g are assumed to be smooth functions of their arguments. The ODE has a very large family of symmetries. (Perhaps surprisingly, so does every first-order ODE.)

We were able to use the known general solution of (1.1) to derive (1.2), which led to the result (1.7). However, we could also have obtained this result directly from (1.2), as follows. On the solution curves, y is a function of x, and hence $\hat{x}(x, y)$ and $\hat{y}(x, y)$ may be regarded as functions of x. Then, by the chain rule, (1.2) can be rewritten as

$$\frac{d\hat{y}}{d\hat{x}} = \frac{D_x\hat{y}}{D_x\hat{x}} = 0 \qquad \text{when} \quad \frac{dy}{dx} = 0,$$

where D_x denotes the *total derivative* with respect to x:

$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots. \tag{1.8}$$

(The following notation is used throughout the book: ∂_x denotes $\frac{\partial}{\partial x}$, etc; y' denotes $\frac{dy}{dx}$, etc.) Therefore (1.2) amounts to

$$\frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y} = 0 \qquad \text{when} \quad y' = 0,$$

that is,

$$\frac{\hat{y}_x}{\hat{x}_x} = 0.$$

Hence (1.7) holds. The advantage of using the symmetry condition in the form (1.2) is that one can obtain information about the symmetries without having to know the solution of the differential equation in advance. This observation is fundamental, for it suggests that it might be possible to find symmetries of a given differential equation whose solution is unknown.

1.3 The Symmetry Condition for First-Order ODEs

The symmetries of $y' = 0$ are easily visualized, because the solution curves are parallel lines. It may not be possible to find symmetries of a complicated first-order ODE by looking at a picture of its solution curves. Nevertheless, the symmetry condition requires that any symmetry maps the set of solution curves in the (x, y) plane to an identical set of curves in the $(\hat{x}, \hat{y})$ plane. Consider a first-order ODE,

$$\frac{dy}{dx} = \omega(x, y). \tag{1.9}$$

(For simplicity, we shall restrict attention to regions of the plane in which ω is a smooth function of its arguments.) The symmetry condition for (1.9) is

$$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}) \qquad \text{when} \quad \frac{dy}{dx} = \omega(x, y). \tag{1.10}$$

As before, we regard y as a function of x (and a constant of integration) on the solution curves. Then (1.10) yields

$$\frac{D_x\hat{y}}{D_x\hat{x}} = \frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y} = \omega(\hat{x}, \hat{y}) \qquad \text{when} \quad \frac{dy}{dx} = \omega(x, y).$$

Therefore the symmetry condition for the first-order ODE (1.9) is equivalent to the constraint

$$\frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}), \tag{1.11}$$

together with the requirement that the mapping should be a diffeomorphism. It may be possible to determine some or all of the symmetries of a given ODE from (1.11). One approach is use an *ansatz*, that is, to look for a symmetry of a particular form.

Example 1.1 Consider the ODE

$$\frac{dy}{dx} = y. \tag{1.12}$$

The constraint (1.11) implies that every symmetry of (1.12) satisfies the partial differential equation (PDE)

$$\frac{\hat{y}_x + y\hat{y}_y}{\hat{x}_x + y\hat{x}_y} = \hat{y}.$$

Rather than trying to find the general solution of this PDE, let us see whether or not there are symmetries that satisfy a simple ansatz. For example, are there any symmetries mapping y to itself? If so, then

$$(\hat{x}, \hat{y}) = (\hat{x}(x, y), y),$$

and the constraint (1.11) reduces to

$$\frac{y}{\hat{x}_x + y\hat{x}_y} = y.$$

Therefore (taking (1.3) into account),

$$\hat{x}_x + y\hat{x}_y = 1, \qquad \hat{x}_x \neq 0.$$

There are many symmetries of this type; the simplest are the Lie symmetries

$$(\hat{x}, \hat{y}) = (x + \varepsilon, y), \qquad \varepsilon \in \mathbb{R}. \tag{1.13}$$

Earlier, we found that translations in the x direction are trivial symmetries of $y' = 0$; are they also trivial symmetries of (1.12)? The general solution of (1.12) is easily found; it is

$$y = c_1 e^x, \qquad c_1 \in \mathbb{R}.$$

A translation (1.13) maps the solution curve corresponding to a particular value of c_1 to the curve

$$\hat{y} = y = c_1 e^x = c_1 e^{\hat{x}-\varepsilon} = c_2 e^{\hat{x}}, \qquad \text{where} \quad c_2 = c_1 e^{-\varepsilon}.$$

Therefore translations in the x direction are nontrivial symmetries of (1.12), because (generally) $c_2 \neq c_1$. (Of course, $\varepsilon = 0$ necessarily gives a trivial symmetry.) Interestingly, one solution curve *is* mapped to itself by every translation, namely $y = 0$. Curves that are mapped to themselves by a symmetry are said to be *invariant* under the symmetry. The solution $y = 0$ partitions the set of solution curves $y = c_1 e^x$, as shown in Fig. 1.4. The translational symmetries

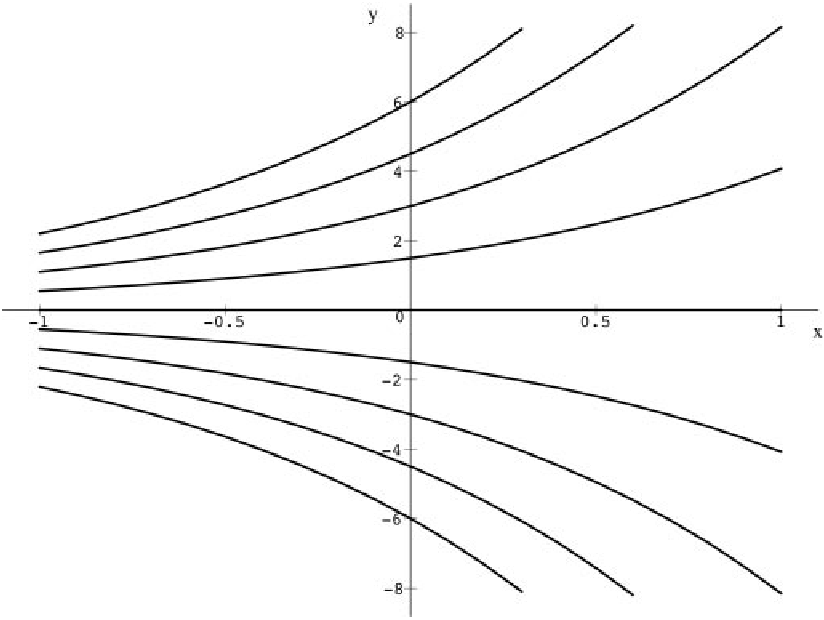

Fig. 1.4. Solutions of $y' = y$.

(1.13) are unable to map solutions with $c_1 > 0$ to solutions with $c_1 < 0$. However, the ODE does have symmetries that exchange the solutions in the upper and lower half-planes. One such symmetry is

$$(\hat{x}, \hat{y}) = (x, -y);$$

this is a discrete symmetry.

So far, we have looked at symmetries of very simple ODEs, but one strength of symmetry methods is that they are applicable to almost any ODE. Here are some more complicated examples.

Example 1.2 The Riccati equation

$$\frac{dy}{dx} = \frac{y+1}{x} + \frac{y^2}{x^3} \tag{1.14}$$

seems complicated, but its general solution is quite simple (as we shall see in the next chapter). The symmetries of this ODE include a one-parameter Lie group of inversions,

$$(\hat{x}, \hat{y}) = \left(\frac{x}{1-\varepsilon x}, \frac{y}{1-\varepsilon x} \right). \tag{1.15}$$

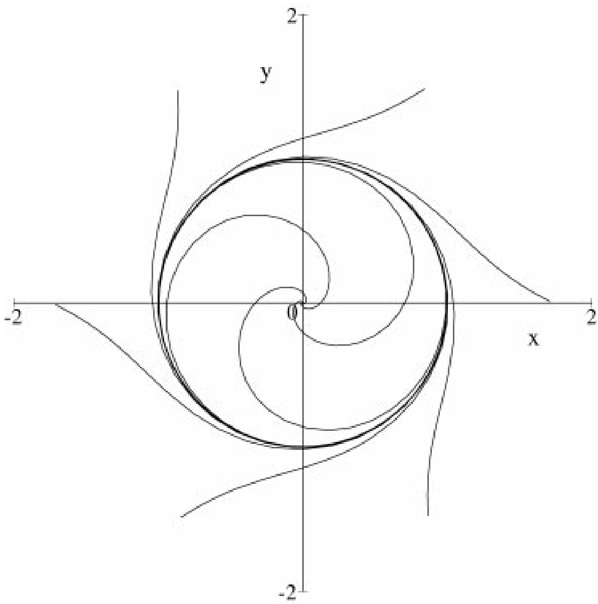

Fig. 1.5. Solutions of (1.16).

To prove this, simply substitute (1.15) into the symmetry condition (1.11), with ω defined by the right-hand side of (1.14). The inversions are our first example of a Lie group of symmetries that is not well defined for all real ε. (The radius of convergence of the Taylor series about $\varepsilon = 0$ is $1/|x|$.)

Example 1.3 Consider the ODE

$$\frac{dy}{dx} = \frac{y^3 + x^2 y - y - x}{xy^2 + x^3 + y - x}. \tag{1.16}$$

The set of solution curves is sketched in Fig. 1.5, which suggests that rotations about the origin are symmetries. It is left to the reader to check that the rotations

$$(\hat{x}, \hat{y}) = (x \cos\varepsilon - y \sin\varepsilon,\ x \sin\varepsilon + y \cos\varepsilon) \tag{1.17}$$

form a one-parameter Lie group of symmetries of (1.16).

1.4 Lie Symmetries Solve First-Order ODEs

The title of this section comes from the following rather surprising result. Suppose that we are able to find a nontrivial one-parameter Lie group of symmetries of a first-order ODE, (1.9). Then the Lie group can be used to determine the general solution of the ODE. This result is an indication of the usefulness of Lie symmetries; it is entirely independent of the function $\omega(x, y)$. The main ideas leading to the result are outlined below, and a more detailed discussion follows in the next chapter.

First, suppose that the symmetries of (1.9) include the Lie group of translations in the y direction,

$$(\hat{x}, \hat{y}) = (x, y + \varepsilon). \tag{1.18}$$

Then the symmetry condition (1.11) reduces to

$$\omega(x, y) = \omega(x, y + \varepsilon), \tag{1.19}$$

for all ε in some neighbourhood of zero. Differentiating (1.19) with respect to ε at $\varepsilon = 0$ leads to the result

$$\omega_y(x, y) = 0.$$

Therefore the most general ODE whose symmetries include the Lie group of translations (1.18) is of the form

$$\frac{dy}{dx} = \omega(x).$$

This ODE can be solved immediately: the general solution is

$$y = \int \omega(x)\, dx + c. \tag{1.20}$$

(We shall regard a differential equation as being solved if all that remains is to carry out *quadrature*, i.e., to evaluate an integral.) The particular solution corresponding to $c = 0$ is mapped by the translation to the solution

$$\hat{y} = \int \omega(x)\, dx + \varepsilon = \int \omega(\hat{x})\, d\hat{x} + \varepsilon,$$

which is the solution corresponding to $c = \varepsilon$. So by using the one-parameter Lie group, we obtain the general solution from one particular solution. The Lie group acts on the set of solution curves by changing the constant of integration.

Clearly, every first-order ODE with the Lie group of translations (1.18) is easily solved. Is the same true for ODEs with other one-parameter Lie groups? Consider the rotationally symmetric ODE (1.16), depicted in Fig. 1.5. It is natural to rewrite the ODE in terms of polar coordinates (r, θ), where

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

We obtain a far simpler ODE,

$$\frac{dr}{d\theta} = r(1 - r^2), \tag{1.21}$$

which is immediately integrable. The one-parameter Lie group of rotations (1.17), rewritten in polar coordinates, becomes

$$(\hat{r},\ \hat{\theta}) = (r,\ \theta + \varepsilon).$$

In the new coordinates, the rotational symmetries become translations in θ, so the ODE is easily solved.

The same idea works for all one-parameter Lie groups. In a suitable coordinate system, the symmetries parametrized by ε sufficiently close to zero are equivalent to translations (except at fixed points). One problem remains: what is the "suitable" coordinate system? For instance, the appropriate coordinate system for the ODE (1.14) is not obvious. It turns out that the Lie group itself holds the solution to this problem, as we shall see in the next chapter.

Exercises

1.1 Sketch the set of solutions of the ODE

$$\frac{dy}{dx} = \frac{y}{x}.$$

How many different kinds of symmetries can you identify?

1.2 Show that the transformation defined by

$$(\hat{x}, \hat{y}) = (e^{\varepsilon}x, y)$$

is a symmetry of

$$\frac{dy}{dx} = \frac{1 - y^2}{x}$$

for all $\varepsilon \in \mathbb{R}$. Describe these symmetries geometrically; how do they transform the solutions of the ODE?

1.3 Verify that the rotations (1.17) are symmetries of the ODE (1.16).

1.4 Determine the value of α for which

$$(\hat{x}, \hat{y}) = (x + 2\varepsilon, ye^{\alpha\varepsilon})$$

is a symmetry of

$$\frac{dy}{dx} = y^2 e^{-x} + y + e^x$$

for all $\varepsilon \in \mathbb{R}$.

1.5 Show that, for every $\varepsilon \in \mathbb{R}$,

$$(\hat{x}, \hat{y}) = \left(x, y + \varepsilon \ \exp\left\{ \int F(x)\, dx \right\} \right)$$

is a symmetry of the general first-order linear ODE

$$\frac{dy}{dx} = F(x)y + G(x).$$

Explain the connection between these symmetries and the linear superposition principle.