STOCHASTIC DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS AND THE CHANGE OF MEASURE METHOD

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1. Introduction

The definition of several types of stochastic integrals for anticipating integrands put the basis for the development, in recent years, of an anticipating stochastic calculus. It is natural to consider, as an application, some problems that can be stated formally as stochastic differential equations, but that cannot have a sense within the theory of non-anticipating stochastic integrals. For example, this is the case if we impose to an s.d.e. an initial condition which is not independent of the driving process, or if we prescribe boundary conditions for the solution.

In this paper, we will try to survey the work already done concerning s.d.e. with boundary conditions, and to explain in some detail a method based in transformations and change of measure in Wiener space. An alternative approach is sketched in the last Section.

Transformations on Wiener space provide a natural method, among others, to prove existence and uniqueness results for nonlinear equations. At the same time, a Girsanov type theorem for not necessarily adapted transformations allows to study properties of the laws of the solutions from properties of the solution to an associated linear equation. A natural first question about these laws is to decide if they satisfy some kind of Markov (or conditional independence) property.

In Section 2, we introduce stochastic differential equations with boundary conditions, the particular instances that have been studied, and the kind of results obtained concerning conditional independence properties of the solutions. The short Section 3 outlines the idea of the method of transformations and change of measure. In Section 4, we recall the necessary elements of Wiener space analysis in order to enounce the Girsanov type theorem we want to apply. In Section 5, we describe the use of suitable transformations to obtain existence and uniqueness results for nonlinear equations. Section 6 is devoted to explain how the change of measure induced by the same transformations allows to derive characterizations of conditional independence properties. Finally, in Section 7 we describe briefly other alternatives to this method and the situations to which they have been applied.

Consider the following general problem:

$$\left.\begin{array}{l} dX_t = f(t,X_t)\,dt + \displaystyle\sum_{i=1}^{k} \sigma_i(t,X_t)\circ dW_t^i \quad , \quad 0\le t\le 1 \\ h(X_0,X_1)=0 \end{array}\right\} \tag{2.1}$$

where $f,\sigma_i\colon [0,1]\times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $h\colon \mathbb{R}^{2d} \longrightarrow \mathbb{R}^d$, and $\{W_t,\ 0\le t\le 1\}$ is a k-dimensional Wiener process ($k\le d$). The customary initial condition for X_0 is replaced by the boundary condition $h(X_0,X_1)=0$.

By a *solution* to (2.1) we mean a d-dimensional continuous process X_t verifying the system

$$\left.\begin{array}{l} X_t = \displaystyle\int_0^t f(s,X_s)\,ds + \sum_{i=1}^{k} \int_0^t \sigma_i(s,X_s)\circ dW_s^i \quad , \quad 0\le t\le 1 \\ h(X_0,X_1)=0 \end{array}\right\} \tag{2.2}$$

But here, unlike the initial-value problem, we cannot expect in general the existence of a solution adapted to the Wiener process, since the boundary condition makes X_0 depend on X_1, which in turn will depend on the whole Wiener process, through the integral equation. Therefore, the stochastic integral in (2.2) has to be understood as an anticipating stochastic integral. With the circle we denote, as usual, the Stratonovich anticipating integral.

As stated in the Introduction, two problems have been tackled concerning such equations: First, of course, the problem of existence and uniqueness of a solution; and secondly, to find sufficient and necessary conditions on the coefficients to have some conditional independence property for the solution.

Which kind of conditional independence property should be expected? The classical Markov Process property will not hold in general because a random variable X_t can hardly make independent the past and the future of the process, given that the first and last variables are linked by the boundary condition.

It turns out that the relevant property to study is the Markov Field property. This is the natural Markov property for random fields, and therefore it is more clearly formulated with a general parameter set:

A random field $\{X_t,\ t\in T\}$, with $T\subset \mathbb{R}^k$, is a *Markov Field* (M.F., for short) if and only if for every Borel and bounded set D (with $\overline{D}\subset T$), the families of random variables $\{X_t,\ t\in \overline{D}\}$ and $\{X_t,\ t\in \overline{D^c}\}$ are conditionally independent given $\{X_t,\ t\in \partial D\}$. Here ∂D denotes the boundary of D. It is enough to check this property for open sets D. Translated to the one parameter case, with $T=[0,1]$, the property can be stated as follows: $\{X_t,\ t\in[0,1]\}$ is a Markov Field if and only if the families $\{X_u,\ u\in[s,t]\}$ and $\{X_u,\ u\in]s,t[^c\}$ are independent given X_s and X_t. It is obvious that every Markov Process is a Markov Field, but the converse is not true.

Let us now take a look to the particular equations studied so far, and the kind of results obtained concerning the Markov Field property. Our aim here is only to sketch these results. We refer the reader to the original references for the precise statements. Particularly, we remark that some technical hypothesis on the coefficients of the equations are needed, first of all, to obtain existence and uniqueness theorems, and then, to characterize the Markov Field property.

A) **First order equations**: Equations like (2.1) are first order equations. Within this setting, the first work was done by Ocone and Pardoux [19] (1989), who considered *linear equations*:

$$\left.\begin{array}{c} dX_t = (AX_t + a(t))\,dt + \displaystyle\sum_{i=1}^{k}(B_iX_t + b_i(t)) \circ dW_t^i \quad , \quad 0 \le t \le 1 \\ F_0X_0 + F_1X_1 = F \end{array}\right\} \tag{2.3}$$

(affine drift, diffusion coefficient and boundary condition), where A, $B_1, \ldots, B_k$, F_0, F_1 are $d \times d$-matrices of constants, $F \in \mathbb{R}^d$, and $a(t), b_1(t), \ldots, b_k(t)$ are d-dimensional processes.

Their main result states that each of the following are sufficient conditions to have a M.F.

a) $B_1 = \cdots = B_k = 0$ (Gaussian case).

b) $a = b_1 = \cdots = b_k = 0$ and $\Phi_t \cdot \Phi_s^{-1}$ is a diagonal matrix, $\forall t, s$, where Φ_t is the matrix solution of

$$\left.\begin{array}{c} d\Phi_t = A\Phi_t\,dt + \displaystyle\sum_{i=1}^{k} B_i\Phi_t \circ dW_t^i \quad , \quad 0 \le t \le 1 \\ \Phi_0 = I \end{array}\right\} \tag{2.4}$$

Notice that, in particular, in dimension one and with linear drift and diffusion, the solution is always a M.F. Further results can be given for special forms of the boundary condition (see [19]).

Nualart and Pardoux [16] considered the non-linear equation

$$\left.\begin{array}{c} dX_t = f(X_t)\,dt + \sigma\,dW_t \quad , \quad 0 \le t \le 1 \\ h(X_0, X_1) = 0 \end{array}\right\} \tag{2.5}$$

and proved that, in dimension 1 ($d = 1$), X is a M.F. iff f is affine. For $d > 1$, they showed examples with nonlinear f for which X is a M.F. (or even a Markov Process) and others where X is not a M.F. With certain special structures of the function f one can recover dichotomy results (see Ferrante [8] and Ferrante–Nualart [9]). ing

Donati-Martin [5] took the next step considering a linear diffusion coefficient:

$$\left.\begin{array}{c} dX_t = f(X_t)\,dt + \sigma X_t \circ dW_t \quad , \quad 0 \le t \le 1 \\ F_0X_0 + F_1X_1 = F \end{array}\right\} \tag{2.6}$$

In dimension 1, the solution is a M.F. iff f takes the form $f(x) = Ax + Bx\log x$, with $|B| < 1$.

B) **Second order equations**: In [17], Nualart and Pardoux studied the following second order stochastic differential equation in dimension one, with Dirichlet type boundary conditions:

$$\left.\begin{array}{rl} \ddot{X}_t = f(X_t, \dot{X}_t) + \dot{W}_t & , \quad 0 \le t \le 1 \\ X_0 = a,\ X_1 = b & \end{array}\right\} \tag{2.7}$$

A solution to (2.7) will be a $\mathcal{C}^1$ process verifying

$$\left.\begin{array}{rl} \dot{X}_t = \dot{X}_0 + \displaystyle\int_0^t f(X_s, \dot{X}_s)\, ds + W_t & , \quad 0 \le t \le 1 \\ X_0 = a,\ X_1 = b & \end{array}\right\} \tag{2.8}$$

In this problem, we cannot expect to have any type of conditional independence property for X, because in a $\mathcal{C}^1$ process the positions X_t do not keep enough information to make independent the past and the future or the interior and exterior of an interval.

However, the two-dimensional stochastic process $\{(X_t, \dot{X}_t),\ t \in [0,1]\}$ is a M.F. iff f is affine. The same result is true if we change Dirichlet to Neumann boundary conditions (see Nualart [14]).

C) **Partial differential equations**: The following parabolic stochastic partial differential equation with periodicity conditions has been considered by Nualart and Pardoux [18]:

$$\left.\begin{array}{rl} \dfrac{\partial X_{t,y}}{\partial t} - \dfrac{\partial^2 X_{t,y}}{\partial y^2} = f(X_{t,y}) + \dfrac{\partial^2 W_{t,y}}{\partial t \partial y} & , \quad (t,y) \in [0,1]^2 \\ X(t,0) = X(t,1) = 0 & , \quad t \in [0,1] \\ X(0,y) = X(1,y) & , \quad y \in [0,1] \end{array}\right\} \tag{2.9}$$

where $\dfrac{\partial^2 W}{\partial t \partial y}$ is a space-time White Noise. The condition f affine is again necessary and sufficient for the Markov Field property of the $C_{0,0}([0,1])$-valued process X_t.

Donati-Martin [6], [7], considered the elliptic equation with Dirichlet boundary condition:

$$\left.\begin{array}{rl} \Delta X_t = f(X_t) + \dot{W}_t & , \quad t \in T \\ X_{|\delta T} = 0 & \end{array}\right\} \tag{2.10}$$

where Δ is the Laplacian, T is a bounded domain of $\mathbb{R}^k$ ($k \le 3$), and $\dot{W}$ represents a White Noise in $\mathbb{R}^k$. Here, the condition f affine is equivalent to a slightly weaker conditional independence property: For any Borel

and bounded set D (with $\overline{D} \subset U \subset T$, for some open set U), the σ-fields $\sigma\{X_t,\ t \in \overline{D}\}$ and $\sigma\{X_t,\ t \in \overline{D^c}\}$ are conditionally independent given the σ-field $\bigcap_{\varepsilon>0} \sigma\{X_t,\ t \in (\partial D)_\varepsilon\}$, where $(\partial D)_\varepsilon$ denotes an ε-neighbourhood of ∂D. X is said to be a *Germ Markov Field* (G.M.F.). □

All these results concerning nonlinear equations have been obtained using a common method, which is based in an argument of change of measure in Wiener space. Our aim is to explain this method, and to illustrate it with some examples. In a non-anticipating context, a change of measure for s.d.e. would rely in the celebrated (Cameron–Martin–Maruyama)–Girsanov Theorem. In our case, an extended version of this theorem, essentially due to Ramer [20] and Kusuoka [12], and allowing for anticipating transformations, is the basic tool to use.

3. Idea of the change of measure method

The idea of the change of measure to study nonlinear anticipating s.d.e. is analogous to that of the classical Girsanov theorem for non-anticipating ones. Starting with a nonlinear equation on a probability space $(\Omega, \mathcal{F}, P)$, with solution process X_t, one considers another measure Q on $(\Omega, \mathcal{F})$, and a linear equation conveniently related with the original one. These measure and linear equation should be chosen in such a way that the law of the solution Y_t under Q coincide with the law of X_t under P.

Then, anything we can prove concerning the law of Y_t under Q produces automatically the same result for the law of X_t under P, which is the process we are interested in. In other words, we switch to a simpler process (possibly in explicit form), at the price of dealing with a more complicated measure, given by its Radon–Nikodým derivative with respect to P.

4. Some elements of analysis in Wiener space

Let (B, H, P) be a Wiener space. That is, H is an infinite-dimensional real separable Hilbert space, equipped with the Gauss cylinder measure μ, B is the completion of H with respect to a measurable norm, and P is the extension of μ to a measure on the Borel σ-field of B. P is called the *Wiener measure* on B. On the other hand, given a Banach space B and a Gaussian centered measure P on B, with $\operatorname{supp} P = B$, there exists a unique Hilbert space $H \subset B$ such that (B, H, P) is a Wiener space. We will call H the *Cameron–Martin space* relative to B and P. See for example Kuo [11] for details.

This is the definition of an *abstract Wiener space*. The *classical Wiener space* is the particular instance in which we take as B the space $C_0([0,1])$ of continuous functions vanishing at zero with the supremum norm, H is the

subspace of functions with derivatives in $L^2([0,1])$, and P is the measure induced by a one-dimensional standard Wiener process.

Many interesting mappings and functionals on Wiener space, such as solutions to s.d.e., are not Fréchet differentiable in general; therefore the classical infinite-dimensional calculus is of little help. For this reason, several infinite-dimensional calculi well adapted to these functionals have been introduced. We recall here the definition of derivation and other basic features of the Malliavin infinite-dimensional calculus.

Let E be a real separable Hilbert space, $F: B \to E$ a mapping and $\omega_0 \in B$. We say that F is *H-differentiable* at ω_0 iff there exists $\nabla F(\omega_0) \in \mathcal{L}(H;E)$ (a linear continuous mapping from H into E) such that

$$\lim_{\substack{h\in H \\ ||h||_H \to 0}} \frac{||F(\omega_0+h) - F(\omega_0) - [\nabla F(\omega_0)](h)||_E}{||h||_H} = 0 \quad .$$

Clearly, if F is Fréchet differentiable at ω_0, then F is H-differentiable at ω_0 and $\nabla F(\omega_0)$ coincides with the Fréchet differential restricted to H.

A *smooth E-valued cylinder functional on B* is a mapping $F: B \to E$ of the form

$$F(\omega) = \sum_{j=1}^{m} f_j((\ell_1,\omega),\dots,(\ell_n,\omega))e_j \quad ,$$

where $\ell_1,\dots,\ell_n \in B^*$ (the topological dual of B), $e_j \in E$, and f_j are $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$ with polynomial growth, together with all their derivatives. Denote by $\mathcal{S}(E)$ the set of these functionals.

Since an element $F \in \mathcal{S}(E)$ is clearly Fréchet differentiable on B, its H-differential exists for every ω and

$$\nabla F(\omega) = \sum_{j=1}^{m}\sum_{i=1}^{n} \partial_i f_j((\ell_1,\omega),\dots,(\ell_n,\omega))\ell_i \otimes e_j \quad ,$$

considered as an element of $\mathcal{L}(H;E)$. Of course, $(\ell_i \otimes e_j, h)$ means $(\ell_i,h)\cdot e_j$, and is an element of the algebraic tensor product $H \otimes E$ (after identification of H and H^*). Its completion by the inner product $\langle \ell \otimes e, \ell' \otimes e' \rangle := \langle \ell, \ell' \rangle_H \cdot \langle e, e' \rangle_E$ is the space of Hilbert–Schmidt operators from H to E, which we will denote by the same symbol $H \otimes E$.

Up to this point, we have only taken into account the topological structure of B. Now, using the measure P, one can prove that $\forall F \in \mathcal{S}(E)$, $\forall p \geq 1$, $F \in L^p(B;E)$ and $\nabla F \in L^p(B; H \otimes E)$, and that $\mathcal{S}(E)$ is dense in $L^p(B;E)$ (see Ikeda and Watanabe [13], Remark 8.2). Moreover, the mapping

$$\nabla: L^p(B;E) \longrightarrow L^p(B; H \otimes E) \quad ,$$

with domain $\mathcal{S}(E)$, is closable. Denoting by $\mathbb{D}^{1,p}(E)$ the closure of $\mathcal{S}(E)$ under the graph norm

$$||F||_{\mathbb{D}^{1,p}(E)} := ||F||_{L^p(B;E)} + ||\nabla F||_{L^p(B;H\otimes E)} \quad ,$$

we obtain a continuous mapping $\nabla: \mathbb{D}^{1,p}(E) \to L^p(B; H\otimes E)$, called the *gradient operator*. Recursively, one can define higher order gradient operators ∇^k ($\nabla^k F$ will be an element of $L^p(B; H^{\otimes k} \otimes E)$) and obtain the Sobolev spaces $\mathbb{D}^{k,p}(E)$.

The operator ∇ is local in the following sense: If, for some measurable set A, $F: B \to E$ verifies $F(\omega) = 0$, for a.a. $\omega \in A$, then $\nabla F: B \to H \otimes E$ verifies the same property. This fact justifies the following definition: The random variable $F: B \to E$ belongs to $\mathbb{D}^{k,p}_{\mathrm{loc}}(E)$ iff there exist a sequence $\{B_n\}_{n\in\mathbb{N}}$ of measurable sets converging to B and a sequence $\{F_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{D}^{k,p}(E)$ such that $F_n = F$ on B_n. For $F \in \mathbb{D}^{k,p}_{\mathrm{loc}}(E)$, the gradient ∇F is defined as $\nabla F(\omega) = \nabla F_n(\omega)$, if $\omega \in B_n$.

The following different concept of differentiability will be used in the Theorem below.

Definition. Let $F: B \to H$ be a random variable with values in the Cameron–Martin space H. The mapping F is H–$\mathcal{C}^1$ if for all $\omega \in B$, there exists a Hilbert–Schmidt operator $\mathcal{K}(\omega)$ such that

1) $||F(\omega + h) - F(\omega) - [\mathcal{K}(\omega)](h)||_H = o(||h||_H)$, as $||h||_H \to 0$, a.s.

2) The mapping $h \mapsto \mathcal{K}(\omega + h)$ from H to $H \otimes H$ is continuous, a.s. □

If F is H–$\mathcal{C}^1$, then $F \in \mathbb{D}^{1,2}_{\mathrm{loc}}(H)$ (see Kusuoka [12] or Nualart [15]). On the other hand, $F \in \mathbb{D}^{1,2}_{\mathrm{loc}}(H)$ implies that $\nabla F(\omega)$ is Hilbert–Schmidt, a.s. Therefore, $\nabla F(\omega)$ is the only candidate for the operator $\mathcal{K}(\omega)$ in the definition.

For any Hilbert-Schmidt operator $\mathcal{K}$ on a Hilbert space H, its Carleman–Fredholm determinant, denoted by $\det_2(I_H + \mathcal{K})$, is defined by

$$\det_2(I_H + \mathcal{K}) := \prod_{i=1}^{\infty} (1 + \lambda_i) e^{-\lambda_i} \quad ,$$

where $\{\lambda_i\}_{i=1}^{\infty}$ is the family of (complex) eigenvalues of $\mathcal{K}$, counted with their multiplicity. For the properties of this quantity and its role in the theory of integral equations see, for instance, Cochran [4].

The Ramer–Kusuoka Theorem can be stated as follows:

Theorem (Ramer [20], Kusuoka [12]).
Let $F: B \to H$ be an H–$\mathcal{C}^1$ map. Assume:

a) The transformation $T: B \to B$ given by $T(\omega) = \omega + F(\omega)$ is bijective.

b) The operator $I_H + \nabla F(\omega): H \to H$ is invertible, a.s.

Then:

The measure $Q := P \circ T$ (the image probability of the Wiener measure by T^{-1}) is equivalent to P and

$$\frac{dQ}{dP}(\omega) = \left| \det_2(I_H + \nabla F(\omega)) \right| \exp\left\{ -(\nabla^* F)(\omega) - \frac{1}{2}||F(\omega)||_H^2 \right\} \quad ,$$

where ∇^* is the adjoint of the gradient operator, considered here as an unbounded operator $L^2(B) \to L^2(B;H)$. □

Remark. ∇^* enjoys a local property which is similar to that of ∇, and ensures that, for $F \in \mathbb{D}^{1,2}_{\mathrm{loc}}(H)$, $\nabla^* F$ is well-defined. □

Remark. There exist stronger versions of this theorem, but we will not make use of them. Particularly, Üstünel and Zakai ([23] [24]) have obtained representations for the density of Q without hypothesis *a)* and with less regularity on F. □

Sometimes it is useful, for the purpose of representation, to realize the Cameron–Martin space H as an L^2 space. Let $(T, \mathcal{B}, \mu)$ be a separable measure space. Denote $\tilde{H} = L^2(T, \mathcal{B}, \mu)$, and let $i: \tilde{H} \to H$ be an isomorphism. Given a random variable $F: B \to H^{\otimes n}$ (for some $n = 0, 1, 2, \dots$), define $G: B \to \tilde{H}^{\otimes n}$ by the equality $F = i^{\otimes n} \circ G$, where $i^{\otimes n}$ is the natural isomorphism between $\tilde{H}^{\otimes n}$ and $H^{\otimes n}$ induced by i. Define, also,

$$DG(\omega) := (i^{\otimes n})^{-1} \circ \nabla F(\omega) \circ i \quad \in \tilde{H}^{\otimes n+1} \quad .$$

If $F \in \mathbb{D}^{1,2}(H^{\otimes n})$, then clearly $G \in \mathbb{D}^{1,2}(\tilde{H}^{\otimes n})$ and we have $DG \in L^2(B; \tilde{H}^{\otimes n+1}) \simeq L^2(B \times T; \tilde{H}^{\otimes n})$.

The operator

$$D: \mathbb{D}^{1,2}(\tilde{H}^{\otimes n}) \longrightarrow L^2(B \times T; \tilde{H}^{\otimes n}) \quad ,$$

which transforms random variables into processes (indexed by the elements of T), is called the *Malliavin derivative operator*.

The statement of the Ramer–Kusuoka Theorem can be translated, using the Malliavin derivative, into a form which is usually more amenable to computations. Let $F: B \to H$, as in the Theorem. Then $DG(\omega) = i^{-1} \circ \nabla F(\omega) \circ i$ is a Hilbert–Schmidt operator on $\tilde{H}$, which has the same eigenvalues of $\nabla F(\omega)$, and the invertibility of $I_H + \nabla F(\omega)$ is equivalent to that of $I_{\tilde{H}} + DG(\omega)$.

Assume now $v: B \to \mathbb{R}$, $v \in L^2(B)$. In this case, $Dv(\omega) = \nabla v(\omega) \circ i$. If we denote by j the isomorphism between $L^2(B; \tilde{H})$ and $L^2(B; H)$ given by $j(F)(\omega) = i(F(\omega))$, we can write $D = j^{-1} \circ \nabla$ (as unbounded operators on $L^2(B)$) and the relation $D^* = \nabla^* \circ j$ holds for the adjoints. The adjoint of the Malliavin derivative operator D is usually denoted by δ and is called the *Skorohod integral*. We have then, for $F: B \to H$ and $F = i \circ G$,

$$\delta G = \nabla^* F \quad .$$

The formula for the density of Q with respect to P becomes

$$\frac{dQ}{dP}(\omega) = \left|\det_2(I_{\tilde{H}} + DG(\omega))\right| \exp\left\{-(\delta G)(\omega) - \frac{1}{2}||G(\omega)||^2_{\tilde{H}}\right\} \quad . \tag{4.1}$$

It is convenient to keep in mind both interpretations of $DG(\omega)$. In fact, the process $\{D_t G(\omega),\ t \in T\}$ is the kernel of the integral operator $DG(\omega)$:

$$[DG(\omega)](h) = \int_T D_t G(\omega) h(t)\, dt, \quad h \in \tilde{H} \quad .$$

In the case of the classical Wiener space, taking $T = [0,1]$ with the Lebesgue measure, we have $i(h) = \int_0^{\cdot} h(t)\, dt$.

5. Transformations. Existence and Uniqueness of solutions

As stated before, transformations in Wiener space provide a natural method to achieve existence and uniqueness results for nonlinear s.d.e. We are going to describe here this procedure, and to apply it to some concrete examples. Another natural approach involves the use of stochastic flows (see, for instance, [5], Theorems 3.1 and 4.1).

Suppose we have a nonlinear equation of the following form:

$$\left.\begin{aligned} p(D)X_t = f(X_t) + \sigma(X_t) \circ \dot{W}_t \quad , \quad t \in T \subset \mathbb{R}^k \\ h(X_{|_{\delta T}}) = 0 \end{aligned}\right\} \tag{N}$$

where $p(D)$ is a linear differential operator with constant coefficients, and we assume that σ is linear or constant. For simplicity of notation we make f, σ and h depend only on X and not on any of its derivatives (cf. equation (2.7)) or the time parameter t, but this is also possible. We assume also that the dimension is equal to one.

$\dot{W}_t$ represents a White Noise. That means: $\dot{W}$ is a centered Gaussian family $\{\dot{W}(A),\ A \in \mathcal{B}(\mathbb{R}^k)\}$ with covariance $E[\dot{W}(A)\dot{W}(B)]$ equal to the Lebesgue measure of $A \cap B$. Its (random) distribution function W_t (normalized with $W_0 = 0$) is a standard Wiener process with k-dimensional parameter. We can (and shall) assume that the probability space $(\Omega, \mathcal{F}, P)$ in which we are working is the canonical space of the Wiener process. Then $W_t(\omega) = \omega(t),\ t \in T$.

Step 1: We can associate to (N) the linear equation

$$\left.\begin{aligned} p(D)Y_t = \alpha Y_t + \sigma(Y_t) \circ \dot{W}_t \quad , \quad t \in T \\ h(Y_{|_{\delta T}}) = 0 \end{aligned}\right\} \tag{L}$$

for some convenient $\alpha \in \mathbb{R}$ (which very often can be taken to be zero). Usually, it is not difficult to find some explicit expression for the solution Y_t.

Step 2: Denote by Σ the set of trajectories of the process Y_t. We must identify this set and check that the mapping from Ω into Σ, defined by $\omega \mapsto Y(\omega)$, is bijective.

Step 3: Define a transformation $T\colon \Omega \to \Omega$ by $T(\omega) = \omega + F(\omega)$, with $F\colon \Omega \to H$. We must choose F in order to have the following:

a) If $Z\colon \Omega \to \Omega$ satisfies $T(Z(\omega)) = \omega$, then $X(\omega) := Y(Z(\omega))$ solves (N).

b) If X solves (N), then there exists such a Z.

F is usually found by inspection or by a formal manipulation of (N) and (L). The following fact is immediate:

Proposition. If T is exhaustive, there exists a solution to (N) whose paths belong to Σ. If, moreover, T is injective, the solution is unique, *within* the class of processes with paths in Σ. □

Example 1. Let us consider first an equation of the type (2.5):

$$\left.\begin{array}{c} \dot{X}_t = f(X_t) + \sigma \dot{W}_t \quad , \quad 0 \le t \le 1 \\ h(X_0, X_1) = 0 \end{array}\right\} \tag{5.1}$$

We assume here $d = 1$ and $\sigma > 0$. The linearized equation (L) is

$$\left.\begin{array}{c} \dot{Y}_t = \alpha Y_t + \sigma \dot{W}_t \quad , \quad 0 \le t \le 1 \\ h(Y_0, Y_1) = 0 \end{array}\right\} \tag{5.2}$$

The solution with initial condition Y_0 can be easily computed and (5.2) becomes equivalent to the system

$$\left.\begin{array}{c} Y_t = e^{\alpha t}\big(Y_0 + \displaystyle\int_0^t \sigma e^{-\alpha s}\, dW_s\big) \quad , \quad 0 \le t \le 1 \\ h\big(Y_0, e^{\alpha}\big(Y_0 + \displaystyle\int_0^1 \sigma e^{-\alpha s}\, dW_s\big)\big) = 0 \end{array}\right\} \tag{5.3}$$

Taking into account that the random variable $\int_0^1 e^{-\alpha s}\, dW_s$ is absolutely continuous and has the whole real line as support, (5.3) will have a solution iff $\forall z \in \mathbb{R}$ (a.e.), $h(y, e^{\alpha} y + z) = 0$ has a solution $y = g(z)$. And this happens when $h(X_0, X_1) = 0$ can be written as $X_0 = g(X_1 - e^{\alpha} X_0)$. Thus, this should be the case to take advantage of considering equation (L). For example, choosing α properly, the case of affine boundary conditions ($aX_0 + bX_1 + c = 0$) is fully covered. But observe also that with periodicity conditions ($X_0 = X_1$) we cannot take the simplest equation ($\alpha = 0$).

The solution to (L) is then

$$Y_t = e^{\alpha t}\big[g\big(e^{\alpha} \int_0^1 \sigma e^{-\alpha s}\, dW_s\big) + \int_0^t \sigma e^{-\alpha s}\, dW_s\big] \quad .$$