PART I

Instability in economic theory

Cambridge University Press
978-0-521-47141-1 - Nonlinear Dynamics and Economics : Proceedings of the Tenth International Symposium in Economic Theory and Econometrics
William A. Barnett, Alan P. Kirman and Mark Salmon
Excerpt
More information

CHAPTER 1

Chaotic dynamics in overlapping generations models with production

Alfredo Medio and Giorgio Negroni

1 Introduction

Over the past decade there has been an increasing interest in the possibility of cyclical and chaotic behavior in perfectly competitive economies. Particular attention has been given to the overlapping generations (OLG) models, with or without production [see, e.g., the contributions of Benhabib and Day (1982), Grandmont (1985), Reichlin (1986), Benhabib and Laroque (1988) and Jullien (1988)].

In this paper we are concerned with the possibility of chaotic dynamics in a two-periods OLG model with production. To the best of our knowledge, in this class of models complex behavior has been found only in the case of *backward perfect foresight*[1] (see e.g., Medio (1992) for the case of a nonmonetary economic with a Leontief technology, and Jullien (1988) for the case of a monetary economy with a production technology that allows for some substitution between factors).

As Woodford (1990) pointed out, this is a rather disturbing occurrence but, as we prove here, it is by no means the only possibility. In fact, in Medio's model, economic agents only work when young and consume only when old, whereas Jullien assumes an exogenously given labor supply. If we modify these restrictive assumptions by endogenizing labor-supply decisions and allowing agents to consume *also* when young, then it is not too hard to overcome the difficulty and to prove the existence of chaotic forward dynamics. As we show in Section 2, the possibility of defining forward dynamics crucially depends on the choice of the utility function for consumption in the second period, whereas the occurrence of complex dynamics also depends on the interaction of the utility function for consumption in the first period and the production function.

[1] See Benhabib and Day (1982) for the possibility of chaotic dynamics with forward perfect foresight in a pure consumption economy.

We consider two classes of utility functions for the first period, labeled CARA and CRRA.[2] Forward dynamics requires that utility function for the second period be always of the CRRA type.

We also consider the two classes of technologies: the linear Leontief and the CES. Combining the technologies and the first-period utility functions, we get four kinds of economies, which we label CARAL(Leontief), CRRAL(Leontief), CARACES, and CRRACES. We prove that each of them admits some combinations of parameters that generate cyclical or chaotic evolutions.

The paper is organized as follows: in Section 2, we discuss the origin of backward perfect foresight in OLG models with production of the traditional type. In Section 3, we introduce our model of intertemporal choice, and in Section 4, we consider the production side of the model in the case of a Leontief technology. In Sections 5 and 6, we present our results, respectively, for CRRAL and CARAL economies. In Section 7, we introduce a CES technology; in Sections 8 and 9, respectively, we analyze the CRRACES and the CARACES economies. We present analytical results whenever it is possible and numerical simulations of the more interesting occurrences. Some economic explanations of complex dynamics are left to Section 10.

2 Backward and forward dynamics: Some preliminary results

In the simpler and most common variation of the OLG model, agents live for two period: they work when young and consume when old. Let w_t and R_{t+1} be, respectively, the real wage rate and the real interest rate; also let $u(c_{t+1})$ and $v(l_t)$ denote the utility of consumption in the second period and the disutility of labor in the first period, respectively. The problem faced by the young (representative) agent at the beginning of period t is thus to choose c_{t+1} and l_t that maximize $[u(c_{t+1}) - v(l_t)]$ subject to the constraints: $k_t = w_t l_t$ and $c_{t+1} = R_{t+1} k_t$.

In this case, from the first-order conditions we get

$$w_t R_{t+1} u'(c_{t+1}) - v'(l_t) = 0.$$

Note, that the first-order conditions may be written as $c_{t+1}u'(c_{t+1}) - l_t v'(l_t) = 0$ from which, putting $\mathcal{U}(c_{t+1}) = c_{t+1}u'(c_{t+1})$ and $\mathcal{V}(l_t) = l_t v'(l_t)$, we obtain $\mathcal{U}(c_{t+1}) - \mathcal{V}(l_t) = 0$. Now, this equation implicitly defines forward perfect

[2] CRRA stands for constant relative risk aversion and has the general functional for $u(c) = \alpha^{-1}c^{\alpha}, 0 < \alpha < 1$; CARA stands for constant absolute risk aversion and has the form $u(c) = -re^{-c}, r > 0$.

foresight dynamics if $\mathcal{U}$ is invertible. If we denote the relative degrees of risk aversion pertaining to the consumption and labor-supply functions by $\rho_2 = -c_{t+1}u''(c_{t+1})/u'(c_{t+1}) > 0$, and $\rho_l = l_t v''(l_t)/v'(l_t) > 0$ respectively, we see that because $\mathcal{U}'(c_{t+1}) = u'[1-\rho_2]$ and $\mathcal{V}'(l_t) = v'[1+\rho_l]$, $\mathcal{U}$ is invertible if the sign of $1-\rho_2$ does not change. This condition, in turn, implies that saving is a monotonic function of the real interest rate.[3]

Reichlin (1986) assumed precisely this condition, and showed that, in the case of a Leontief technology, for $\rho_2 < 1$ the stationary state loses its stability through a Neimark[4] bifurcation when $b \geq 2$, where b represents the output-capital ratio. On the other hand, Reichlin also showed that, with a production technology with variable proportions, a Neimark bifurcation is possible, provided that the elasticity substitution between capital and labor at the steady state is less that the share of capital income.

Medio (1992) used the same framework as Reichlin to prove the existence of backward perfect-foresight chaotic dynamics, for the case of a Leontief technology. The author here assumed a utility function of the CARA type for second-period consumption, and a disutility function of the CRRA type for current labor supply.

3 Forward dynamics: The basic model

Let us consider an economy composed of two overlapping generations. The members of each generation live for two periods (youth and old age) and work only when young, but consume in both periods of life. To simplify the analysis, we consider a real economy in which there is only one commodity (e.g., corn) that can be consumed or used in the production process.

Let us assume that the overall utility function is time separable. The utility derived from consumption in the first and second period, and the disutility of labor in the first period, are denoted by $u_1(c_t)$, $u_2(c_{t+1})$, and $v(l_t)$, respectively. We further assume that the functions u_1, u_2 and v are continuous on $[0, +\infty)$ and that, for $c, l > 0$, they satisfy the following conditions: $u_i'(c) > 0$, $u_i''(c) < 0$, for $i = 1, 2$, and $v'(l) > 0$, $v''(l) > 0$.

[3] On the other hand, because $\mathcal{V}' = v'(1+\rho_l) > 0$ always, it follows that it is always possible to derive backward solutions. Indicating with s_t the young agent's saving at the beginning of period t, from the budget constraint we get: $c_{t+1} = R_{t+1}s_t$ and $l_t = s_t/w_t$. Substituting these variables into the first-order conditions, we get $\phi(s_t, R_{t+1}, w_t) = 0$. Now because $\partial\phi/\partial R_{t+1} = w_t(u' + R_{t+1}s_t u'')$, and $\partial\phi/\partial w_t = R_{t+1}u' + s_t v'' w_t^{-2}$, it can be seen immediately that $\partial\phi/\partial w_t > 0$, always; on the contrary, $\partial\phi/\partial R_{t+1}$ is positive if $u' + c_{t+1}u'' > 0$, that is if $1 > \rho_2$

[4] We use the term Neimark bifurcation instead of Hopf bifurcation for discrete-time dynamical systems, because the basic results in this area were not found by Hopf but by Neimark (and Sarker). Moreover, using the same name for two totally different phenomena is confusing. A broad definition of this bifurcation, and some references, are given in Section 5.

The problem faced by the young (representative) agent at the beginning of period t is to choose c_t, c_{t+1}, and l_t that solve the following program:

$$\begin{aligned} \max \quad & [u_1(c_t) + u_2(c_{t+1}) - v(l_t)] \\ \text{s.t.} \quad & s_t k_t \le w_t l_t - c_t \\ & c_{t+1} \le R_{t+1} k_t \\ & c_t, c_{t+1}, k_t, l_t > 0. \end{aligned}$$

The first-order conditions are:

$$u_1'(c_t) - R_{t+1} u_2'(c_{t+1}) = 0 \tag{1.1}$$

$$u_2'(c_{t+1}) R_{t+1} w_t - v'(l_t) = 0 \tag{1.2}$$

$$c_{t+1} = R_{t+1}(w_t l_t - c_t). \tag{1.3}$$

Solving equation (1.3) for $R_{t+1} w_t$, and substituting into equation (1.2) we get the following equation:

$$u_2'(c_{t+1}) c_{t+1} + u_2'(c_{t+1}) R_{t+1} c_t - v'(l_t) l_t = 0. \tag{1.4}$$

From equation (1.1) we also get $R_{t+1} = u_1'(c_t)/u_2'(c_{t+1})$. Substituting this expression into equation (1.4), we get

$$u_1'(c_t) c_t + u_2'(c_{t+1}) c_{t+1} - v'(l_t) l_t = 0.$$

Let us now define the following new functions: $\mathcal{U}_1(c_t) \equiv u_1'(c_t) c_t$, $\mathcal{U}_2(c_{t+1}) \equiv u_2'(c_{t+1}) c_{t+1}$, and $\mathcal{V}(l_t) \equiv v'(l_t) l_t$. Then we have

$$\frac{\partial \mathcal{U}_1(c_t)}{\partial c_t} = u_1'(c_t)[1 - \rho_1],$$

$$\frac{\partial \mathcal{U}_2(c_{t+1})}{\partial c_{t+1}} = u_2'(c_{t+1})[1 - \rho_2],$$

$$\frac{\partial \mathcal{V}(l_t)}{\partial l_t} = v'(l_t)[1 + \rho_l],$$

where $\rho_j (j = 1, 2, l)$ are the degrees of relative risk adversion associated with the relevant (dis)utility functions, as defined above. Using the new functions, the dynamical system derived from the consumer's choice is implicitly defined by the following equation:

$$\mathcal{U}_1(c_t) + \mathcal{U}_2(c_{t+1}) - \mathcal{V}(l_t) = 0. \tag{1.5}$$

To get forward dynamics, we need $\mathcal{U}_2(c_{t+1})$ to be invertible in the relevant domain; this, in turn, requires that over the domain, the sign of $(1 - \rho_2)$ does not change. When this condition is satisfied, we have

$$c_{t+1} = \mathcal{U}_2^{-1}[\mathcal{V}(l_t) - \mathcal{U}_1(c_t)] = h(l_t, c_t). \tag{1.6}$$

In the sequel of the paper we consider the following utility functions:

$$u_1(c_t) = -re^{-c_t} \quad r > 0 \tag{1.7a}$$

$$u_1(c_t) = \frac{1}{\theta}c_t^\theta \qquad 0 < \theta < 1 \tag{1.7b}$$

$$u_2(c_{t+1}) = \frac{1}{\alpha}c_{t+1}^\alpha \quad 0 < \alpha < 1 \tag{1.7c}$$

$$v(l_t) = \frac{1}{\gamma}l_t^\gamma \quad \gamma > 1. \tag{1.7d}$$

Note that equations (1.7b–d) are utility functions of the CRRA type, whose coefficients of relative risk adversion are, respectively, $\rho_1 = 1-\theta$, $\rho_2 = 1-\alpha$, and $\rho_l = \gamma - 1$, and therefore independent of c_t, whereas utility function (7.1a) is of the CARA type, with $\rho_1 = f(c_t) = c_t$. In what follows, we use the labels CARA or CRRA when in the dynamical equation (1.6) there appears a function of type (1.7a) or (1.7b), respectively.

3.1 *Saving function*

Let us now consider equation (1.4). From this we wish to derive the agent's saving function. Let $s_t = w_t l_t - c_t$ denote the saving of the young agent in period t. From this definition and from the budget constraints we get

$$u_1'(w_tl_t - s_t)[w_tl_t - s_t] + u_2'(R_{t+1}s_t)R_{t+1}s_t - v'[(s_t+c_t)/w_t][(s_t+c_t)/w_t] = 0.$$

In general terms, this equation can be written as $\phi(s_t, R_{t+1}, w_t, l_t, c_t) = 0$. Provided that $\partial\phi/\partial s_t \neq 0$, this equation implicitly defines a saving function: $s_t = g(R_{t+1}, w_t, l_t, c_t)$.

The elasticity of saving with respect to the interest rate is

$$\frac{\partial s_t}{\partial R_{t+1}}\frac{R_{t+1}}{s_t} = \frac{1-\rho_2}{1+\rho_2+\rho_l-\rho_1}. \tag{1.8}$$

Note that had we not introduced the possibility of consumption in the first period, the elasticity of saving with respect to the interest rate would have been $(1-\rho_2)(\rho_2+\rho_l)^{-1}$, as in Reichlin (1986). Thus, for a given value of ρ_2 we may get a value of the elasticity of saving with respect to the interest rate higher (or lower) than that of the Reichlin's model, depending on the values of ρ_1 and ρ_l.[5]

[5] In Reichlin's model, saving is an increasing function of the interest rate if $\rho_2 < 1$. In the CRRA case, saving is always an increasing function of R; in the CARA case, however, saving is positively or negatively related to R, according to whether c_t is smaller or larger than $1+\gamma-\alpha$.

Using the functional forms (1.7), the elasticity of saving with respect to the real interest rate becomes

$$\frac{\partial s_t}{\partial R_{t+1}} \frac{R_{t+1}}{s_t} = \frac{\alpha}{1+\gamma-\alpha-c_t}$$

for the CARA case, and

$$\frac{\partial s_t}{\partial R_{t+1}} \frac{R_{t+1}}{s_t} = \frac{\alpha}{\gamma-\alpha+\theta}$$

for the CARA case.

Making use of equations (1.7a,c,d), equation (1.6) becomes

$$c_{t+1} = [l_t^{\gamma} - rc_t e^{-c_t}]^{1/\alpha}. \tag{1.9a}$$

If instead we use equations (1.7b–d), we get

$$c_{t+1} = [l_t^{\gamma} - c_t^{\theta}]^{1/\alpha}. \tag{1.9b}$$

Equation (1.9a), or (1.9b), represents the optimal evolution of consumption, derived from consumer's intertemporal choice of consumption and leisure only. It is, so to speak, the first half of our dynamical system, to complete which we need a second equation that takes the technology side of the system into account.

4 Leontief technology

Let us turn to the production side of the economy. As mentioned before, we consider two different production technologies: linear Leontief and CES.

In the former case, we assume that output in period t, x_t, is produced by current labor and capital invested in the previous period; thus,

$$x_t = \min[al_t, bk_{t-1}], \tag{1.10}$$

where $b > 1$ (to ensure variability of the economy). In this kind of economy, we get $R_t = b(1-w_t)$. Also, the equilibrium condition in the product market yields

$$x_t = k_t + c_t. \tag{1.11}$$

For simplicity, in what follows we put $a = 1$. From the assumption of full employment of capital, we have $x_t = bk_{t-1}$ and, taking into account the equilibrium condition (1.11), we obtain $x_t = b(x_{t-1} - c_{t-1})$. From the assumption of full employment of labor, and remembering that $a = 1$, we have $x_t = l_t$. Hence, moving forward one period, we obtain the second dynamical equation of the model:

$$l_{t+1} = b(l_t - c_t). \tag{1.12}$$

Equations (1.9) and (1.12) represent the evolution of the system that is compatible with intertemporal optimization and equilibrium conditions in a Leontief economy.

5 Chaotic dynamics in a CRRAL economy

Let us now consider the simpler case in which both the utility functions for first- and second-period consumption are of a CRRA type (the labor disutility function is of this type throughout the paper). In this case, we have the following dynamical system:

$$c_{t+1} = (l_t^{\gamma} - c_t^{\theta})^{1/\alpha} \tag{1.13a}$$

$$l_{t+1} = b(l_t - c_t), \tag{1.13b}$$

where $\gamma > 1, b > 1$, and $0 < (\alpha, \theta) < 1$.

The system (1.13a,b) has two equilibria. The first, a trivial one, is $E_1 : (\bar{l} = 0, \bar{c} = 0)$. The second equilibrium cannot be computed explicitly, but we can show that it is unique and strictly positive (In what follows, we assume that $\alpha \geq \theta$. All of the relevant results could be proved analogously for $\alpha < \theta$).

Lemma 1.1: *System (1.13a, b) has a unique positive equilibrium at* E_2 : $(\bar{c} > 1, \bar{l} > 1)$.

Proof: From equation (1.13b), we get $\bar{l} = b(b-1)^{-1}\bar{c}$. From equation (1.13a), we get $1 + \bar{c}^{\theta-\alpha} = \psi\bar{c}^{\gamma-\alpha}$, where $\psi \equiv b^{\gamma}(b-1)^{-\gamma}$. Let us call the left-hand side and the right-hand side of this equation $f(c)$ and $g(c)$, respectively. Observe that, having assumed $\alpha > \theta$, it follows that $\lim_{c\to 0} f(c) = \infty$, and $\lim_{c\to\infty} f(c) = 1$; note also that $f'(c) < 0$ and $f''(c) > 0$. Regarding $g(c)$, we have $g(0) = 0$, $\lim_{c\to\infty} g(c) = \infty$ and $g'(c) > 0$. It follows that $f(c) = g(c)$ has a unique solution and this solution is in the first orthant of the (c, l) plane. □

The content of Lemma 1.1 is represented in Figure 1.1

Let us now analyze the local stability of E_1. Evaluating the Jacobian matrix at E_1, we get

$$J_1 = \begin{bmatrix} 0 & 0 \\ -b & b \end{bmatrix}$$

when $\theta < \alpha$. In this case, from inspection of J_1 we see that the eigenvalues are respectively 0 and $b > 1$. Therefore, E_1 is locally unstable and it is not possible to find oscillatory motion around it. (Note that when $\theta > \alpha$, the first eigenvalue is equal to $-\infty$, whereas the second in still equal to $b > 1$).

Let us now consider local stability of E_2. Evaluating the Jacobian matrix at E_2, we get

$$J_2 = \begin{bmatrix} -\frac{\theta}{\alpha}c^{\theta-\alpha} & \frac{\gamma}{\alpha}\left(\frac{b}{b-1}\right)^{\gamma-1}c^{\gamma-\alpha} \\ -b & b \end{bmatrix}.$$

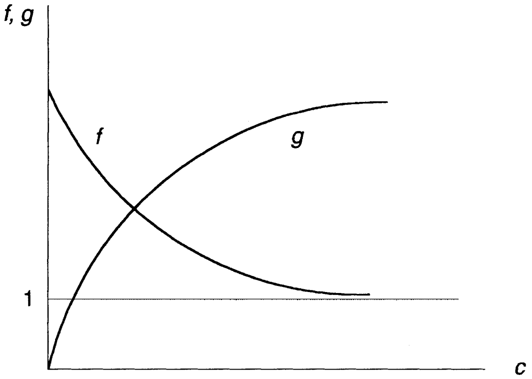

Figure 1.1.

Therefore, we have

$$\text{Tr } J_2 = -\frac{\theta}{\alpha} c^{\theta-\alpha} + b,$$

$$\text{Det } J_2 = (b-1)\frac{\gamma}{\alpha} + [\gamma(b-1) - b\theta]\frac{c^{\theta-\alpha}}{\alpha}.$$

Keeping in mind that stability of equilibrium requires that the following conditions be met

$$\begin{aligned} &\text{(i)} \quad 1 + \text{Tr } J_2 + \text{Det } J_2 > 0, \\ &\text{(ii)} \quad 1 - \text{Tr } J_2 + \text{Det } J_2 > 0, \\ &\text{(iii)} \quad 1 - \text{Det } J_2 > 0, \end{aligned} \tag{1.13c}$$

we can now state the following result:

Proposition 1.1: *Let us consider the equilibrium E_2, of the dynamical system (1.13a,b); E_2 is a stable equilibrium for sufficiently small γ, sufficiently large α, or sufficiently small b.*

Proof: For simplicity, we prove the proposition assuming $\alpha = \theta$, but the result extends in an obvious manner to the case $\alpha > \theta$. In this case, the set of inequalities (1.13c) that governs the stability conditions becomes: (i) $2\gamma(b-1)\alpha^{-1} > 0$; (ii) $2(b-1)(\gamma-\alpha)\alpha^{-1} > 0$; (iii) $2\gamma(b-1)\alpha^{-1} - b - 1 < 0$. Notice that the first two inequalities are always satisfied, because γ and b are both larger than one. The third inequality is obviously satisfied if γ is small, α is large, or b is near 1. Therefore, low productivity, low elasticity of consumption utility function, and high elasticity of labor utility function work for stability. □