This book is an introduction to integrability and conformal field theory in two dimensions using quantum groups. The book begins with a brief introduction to S-matrices, spin chains and vertex models as a prelude to the study of Yang–Baxter algebras and the Bethe ansatz. The basic ideas of integrable systems are then introduced, with particular emphasis on vertex and face models. Special attention is given to explaining the underlying mathematical tools, including braid groups, knot invariants and towers of algebras. The book then goes on to give a detailed introduction to quantum groups as a prelude to chapters on integrable models, two-dimensional conformal field theories and super-conformal field theories. The book contains many diagrams and exercises to illustrate key points in the text.

This book will be of use to graduate students and researchers in theoretical physics and applied mathematics interested in integrable systems, string theory and conformal field theory.

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QUANTUM GROUPS IN TWO-DIMENSIONAL PHYSICS

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CÉSAR GÓMEZ
Consejo Superior de Investigaciones Científicas, Madrid

MARTÍ RUIZ-ALTABA
Université de Genève

GERMÁN SIERRA
Consejo Superior de Investigaciones Científicas, Madrid

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A nuestras niñas

Ana, Camila, Laura, Marina, Martina y Pepa

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Preface

Satius est supervacua discere quam nihil
Seneca

This book addresses the need among theoretical physicists and mathematicians for a modern, intuitive and moderately comprehensive introduction to the subject of integrable systems in two dimensions, one plus one or two plus zero. The requisite background for reading this book profitably amounts to elementary quantum field theory and statistical mechanics, in addition to basic group theory. We have tried to present all the material, both standard and new, in modern language and consistent notation.

It is perhaps still premature to evaluate the real physical impact of string theory, but it is certainly true that the current renaissance of two-dimensional physics owes much to the string wave. Traditionally, physics in two dimensions was considered a theoretical laboratory, the realm of toy models. Only after the recent work on string theory did two-dimensional quantum field theories graduate from pedagogical simplifications to serious candidates for the understanding of nature: physics in the purest aristotelian sense.

Independently of how much truth lies within string theory or elsewhere, a beautiful feature of physics in two dimensions is of course its mathematical richness. Astonishingly, almost any branch of mathematics becomes relevant in the study of two-dimensional field theories. The main physical reason for such mathematical inflation is the existence of non-trivial completely integrable two-dimensional field theories. More technically, the wonders of two dimensions have their origin in the powerful artillery of complex analysis. It is remarkable that so much of what we now understand in great generality was already contained in Onsager's solution (1944) of the two-dimensional Ising model. In particular, he discovered the star-triangle relation, now called the Yang–Baxter equation.

At the root of integrability we find a kind of trivial dynamics, described by factorizable S-matrices. This dynamics, implying infinite-dimensional symmetries, takes its most concrete expression in the wonderful Yang–Baxter equation, linking solvable two-dimensional models in statistical mechanics and quantum field theory to knot invariants and quantum groups.

The Yang–Baxter equation was originally formulated as a condition that the basic quantities of the model (be it Boltzmann weights, scattering matrices, or braiding matrices) should satisfy in order for the theory to be solvable. Later on, it was realized that there exists a hidden symmetry underlying the trigonometric and rational solutions to the Yang–Baxter equation. This hidden symmetry is captured by a new concept in mathematics, the quantum group, which unifies the framework of two-dimensional exact models. History teaches us that whenever a new kind of symmetry is discovered, a revolution is knocking at the door of knowledge. In these revolutionary times, new ideas abound and the dust has not yet begun to settle in order to distinguish more clearly what is truly revolutionary from what is fashionable opportunism; we feel nevertheless that rephrasing the Yang–Baxter equation in terms of a symmetry is of enormous epistemological relevance.

Although quantum groups were born from integrability, providing us with an algebraic explanation for the Yang–Baxter equation in terms of symmetry, they constitute such an interesting conceptual breakthrough that the whole subject of integrable models deserves a re-examination in their light. The tender age of quantum groups (about ten years old at the time of this writing) hides somewhat the fact that, albeit in disguise, they had already surfaced earlier in physics and mathematics, for example in the discrete calculus associated with q-numbers. Much work remains to be done in the development of quantum groups, now a coherent foundation upon which fancier towers may be built. A major challenge, for instance, is to understand the elliptic solutions to the Yang–Baxter equation, notably that to the eight-vertex model, in the quantum group language.

Already, the deeper understanding of integrability afforded by quantum groups has allowed the construction of new integrable models. The old and historic models, such as the Heisenberg, the sine–Gordon, or the six-vertex models, have thereby multiplied into quantum group descendants, and the growing family is still not complete. Let us stress that, for the time being, quantum groups have remained confined to two-dimensional physics: either two non-relativistic spatial dimensions (statistical mechanics) or one time and one space (quantum field theory). Higher dimensional applications of quantum groups are perhaps possible though at any event very rare, due perhaps to the difficulty of finding integrable models in dimensions higher than two. Most likely, the quantum group symmetry is intimately tied with two dimensions, and any extension to other dimensions of the quantum group technology will call for a different algebraic structure. However desirable *a priori* these extensions might appear, the perfect uniqueness of strings takes away much of the motivation for looking anywhere else than two dimensions for fundamental structures, and thus the research program around quantum groups acquires even more urgency and appeal.

The above considerations explain the ideology behind this book, which attempts to distill the structure of quantum groups from two-dimensional physics and, conversely, to frame physical questions in a formalism such that quantum groups

may provide us with the answers. Were the various topics in this book not so closely linked by quantum groups, one would dare call our work an interdisciplinary effort between physics and mathematics, but given the environment of prime interest to us it is perhaps best to speak of a physics book with mathematical applications. In the spirit of the Vienna circle, mathematical physics should be considered as synonymous with formalization. Nevertheless, in our days mathematical physics is approaching criticism and shying away from the old-fashioned aim of axiomatization. For a critic, the material consists typically of works of art, and the goal is to find the clues to provide new and unifying points of view. In this sense, this book is closer in spirit to modern criticism than to canonical formalization. Our *corpus* is made out of two-dimensional theoretical physics. The chosen outlook hinges on the symmetry clue: we rely on quantum groups to extend our understanding of symmetries in physics. Recalling the French Anatolian saying that a good critic is somebody who describes their adventures among masterpieces, we can only hope that this book provides the reader with a pleasant tour.

We start with an introduction to integrable vertex models: in chapters 1 and 2 we introduce factorized S-matrices, the Bethe ansatz and the Yang–Baxter equation, along with the basic concepts about quantum groups. Chapter 3 reviews the Bethe ansatz solution to some simple spin chain hamiltonians. It also includes a few words on more general spin chains, and its last sections present boundary effects and Sklyanin's algebra. Chapters 4 and 5 serve to introduce the reader to some mathematical tools not generally known to practicing physicists, while discussing the algebraic Bethe ansatz solution to the eight-vertex model and, thus motivated, the face models. The trigonometric solution to the latter is presented in the general framework of the Temperley–Lieb–Jones algebra, complemented by an appendix on knot theory and another one on finite-dimensional subfactors. Chapter 6, the mid-point of the book, contains most of the necessary mathematical information about quantum groups, both finite and affine. In chapter 7 we turn to the representation theory of quantum groups at roots of unity and present several physical applications thereof, with an emphasis on explicit calculations. Chapter 8 introduces the reader to the universal behavior of second order phase transitions, conformal field theory, and the decoupling of null vectors. Chapter 9 exploits the concepts introduced along the book to study the duality structure of rational conformal field theories. Chapter 10 proceeds to the free field representation of these theories and introduces also the simplest Wess–Zumino models. Finally, in chapter 11, we use the free field realization of rational conformal field theories to discuss their quantum symmetries.

To keep the book to a reasonable size, we have been forced to make some painful choices. Quite a few relevant topics have been discussed only very briefly – we hope that the appendices and the exercises will fill these gaps to some extent.

Each of the chapters ends with a very brief bibliographical overview for further reading. We have not made any attempt at comprehensiveness, and we cite only

essential major works that we have actually used. These references should serve the reader as an introduction to the vast literature available.

In the last few years, each of us has given various series of lectures on quantum groups, conformal field theory and integrable models in different places. This volume is an outgrowth of the course which one of us (C.G.) taught at the Troisième Cycle de Physique de la Suisse Romande during the spring of 1990, and it owes much to the feedback provided by a number of eager and inquisitive audiences.

This book was invented, as a non-existing object, by Henri Ruegg. Only thanks to his insistent prodding does it now become part of a reality, as realities occur in a Borges universe.

We have learned this physics from many colleagues. In particular, we wish to mention Luis Alvarez-Gaumé with whom we started the study of conformal field theories and quantum groups several years ago, Alexander Berkovich, who shared with us his knowledge of the Bethe ansatz, and Cupatitzio Ramírez, with whom we developed the contour picture of quantum groups. We would also like to acknowledge interesting scientific discussions with Rodolfo Cuerno, Tetsuji Miwa, Carmen Núñez and Philippe Zaugg.

Finally, we wish to acknowledge the support of the whole Theoretical Physics Department at the University of Geneva, and of the Fonds National Suisse pour la Recherche Scientifique.

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