1

S-matrices, spin chains and vertex models

In classical mechanics, two functions over phase space are said to be in involution if their Poisson bracket vanishes. Since Liouville's time, a dynamical system whose phase space is $2N$ dimensional is called completely integrable if there are N functions, or "hamiltonians", or "charges" in involution. A field theoretic system is called integrable if it possesses an infinity of mutually commuting conserved observables. All these mutually commuting conserved charges or hamiltonians allow us to solve the system exactly, without resorting to approximation schemes. Integrability is an unusual wealth of symmetry we might not think of requiring on realistic physical models. Rather, we should expect the complexity of nature not to be exactly solvable. However, integrability is of epistemological importance: exact solutions allow more perfect understanding. Toy models, of which physicists have been very fond since antiquity, often exist just so that exact and complete solutions can be found, in order to grasp the nature of the phenomenon being modeled. Furthermore, and quite surprisingly, physical systems with an infinite symmetry do exist: any non-linear system with soliton solutions is integrable. We shall be interested in discovering under which circumstances certain kinds of physical systems admit complete integrability, what types of systems these are, and in pointing out the physical roots of such a wonderful property. In the process, we shall have the occasion to use some of the most powerful tools elaborated by workers in mathematics.

Given our present understanding of two-dimensional models, integrability appears as a consequence of very simple dynamics, characterized by factorized scattering matrices. In this first chapter, we shall become acquainted with some of the most striking properties of factorized S-matrices, following Zamolodchikov; we shall then introduce Bethe's classical and beautiful work on the one-dimensional Heisenberg ferromagnet.

1.1 Factorized S-matrix models

Consider the scattering of relativistic massive particles in a $(1+1)$-dimensional spacetime. There is only one spatial dimension (the real line, say), and therefore the ordering of the particles is a well-defined, frame-independent concept. Equivalently, the distinction between left and right is unambiguous; in more spatial dimensions this is never so, and thus we should not expect that the interesting features depending strongly on the ordering of the particles extend to theories in higher dimensions.

It is convenient to use as kinematical variable for each particle a rapidity θ, in terms of which the momentum p^1 and energy p^0 read as follows:

$$p^0 = m \cosh\theta\ , \qquad p^1 = m \sinh\theta \tag{1.1}$$

This parametrization ensures the on shell condition $\vec{p}^{\,2} = (p^0)^2 - (p^1)^2 = m^2$.

Alternatively, we could use the lightcone momenta p and $\overline{p}$,

$$p = p^0 + p^1 = m\,\mathrm{e}^{\theta}\ , \qquad \overline{p} = p^0 - p^1 = m\,\mathrm{e}^{-\theta} \tag{1.2}$$

which transform under a Lorentz boost, $L_\alpha : \theta \to \theta + \alpha$, as

$$p \to p\,\mathrm{e}^{\alpha}\ , \qquad \overline{p} \to \overline{p}\,\mathrm{e}^{-\alpha} \tag{1.3}$$

Quite generally, an irreducible tensor Q_s of the Lorentz group in $1+1$ dimensions is labeled by its spin s according to the rule

$$L_\alpha : Q_s \to \mathrm{e}^{s\alpha} Q_s \tag{1.4}$$

so that p is of spin 1 and its parity conjugate $\overline{p}$ is of spin -1.

Suppose that Q_s is a local conserved quantity of spin $s > 0$ (the case $s < 0$ is obtained by a parity transformation) in a scattering process involving n particles A_i $(i = 1, \ldots, n)$ with masses m_i. On a one-particle state $|A_i(\theta)\rangle$ of rapidity θ, the operator Q_s acts as

$$Q_s\,|A_i(\theta)\rangle \sim p^s\,|A_i(\theta)\rangle \tag{1.5}$$

Since Q_s is local and conserved by assumption, the scattering process must satisfy

$$\sum_{i\in\{\text{in}\}} p_i^s = \sum_{f\in\{\text{out}\}} p_f^s \tag{1.6}$$

If the theory happens to be parity invariant, then we also have

$$\sum_{i\in\{\text{in}\}} \overline{p}_i^s = \sum_{f\in\{\text{out}\}} \overline{p}_f^s \tag{1.7}$$

Setting $s = 1$ in (1.6) and (1.7), we recover the usual energy and momentum conservation laws of a relativistic theory.

We are interested in theories with conserved quantities of higher spin (i.e. $|s| > 1$), leading to the conservation laws (1.6) and (1.7). In fact, integrability is synonymous with an infinity of such conserved higher spin quantities.

The physical behavior of integrable systems is quite remarkable. For instance, if (1.6) and (1.7) hold for an infinity of different spins s, it follows immediately that the incoming and outgoing momenta must be the same:

$$\left\{p_i^\mu\ ; i \in \text{in}\right\} = \left\{p_f^\mu\ ; f \in \text{out}\right\} \tag{1.8}$$

This means that no particle production or annihilation may ever occur.

Also, particles with equal mass may reshuffle their momenta among themselves in the scattering, but particles with different masses may not. Equivalently, we may say that the momenta are conserved individually and that particles of equal mass may interchange additional internal quantum numbers. If all the incoming particles have different masses, then the only effect of the scattering is a time delay (a phase shift) in the outgoing state with respect to the incoming one.

The infinite symmetry of the physical systems we are describing restricts tremendously the allowed processes, much to our intellectual advantage. As we shall see now, all scattering processes can be understood and pictured as a sequence of two-particle scatterings. This property is called factorizability.

By relativistic invariance, the scattering amplitude between two particles A_i and A_j may only depend on the scalar

$$p_i^\mu p_j^\nu \eta_{\mu\nu} = m_i m_j \cosh\left(\theta_i - \theta_j\right) \tag{1.9}$$

so that, in fact, it may depend only on the rapidity difference $\theta_{ij} = \theta_i - \theta_j$. Using (1.8), the most general form of the basic two-particle S-matrix in terms of which all other S-matrices will be written is thus

$$S\ |A_i(\theta_1), A_j(\theta_2)\rangle_{\text{in}} = \sum_{k,\ell} S_{ij}^{k\ell}(\theta_{12})\, |A_k(\theta_2), A_\ell(\theta_1)\rangle_{\text{out}} \tag{1.10}$$

In this notation, $|A_i(\theta_1), A_j(\theta_2)\rangle_{\text{in(out)}}$ stands for the initial (respectively, final) state of two incoming (respectively, outgoing) particles of kinds A_i and A_j and rapidities θ_1 and θ_2. This elementary process is shown diagrammatically in figure 1.1.

In addition to (1.8), the second crucial feature of a factorizable S-matrix theory, from which such models get their name, is the property of factorizability: the N-particle S-matrix can always be written as the product of $\binom{N}{2}$ two-particle S-matrices.

As in equation (1.10), we choose an initial state of N particles with rapidities $\theta_1 > \theta_2 > \cdots > \theta_N$ arranged in the infinite past in the opposite order, i.e. $x_1 < x_2 < \cdots < x_N$. This presumes simply that no scatterings have occurred before we begin studying the process, i.e. that we have been observing the system long before any particles meet. After the $N(N-1)/2$ pair collisions, the particles

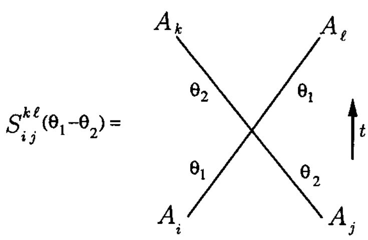

Fig. 1.1. Collision of two particles A_i and A_j with rapidities θ_1 and θ_2 ($\theta_1 > \theta_2$) into particles A_k and A_ℓ.

reach the infinite future ordered along the spatial direction in increasing rapidity. Thus we write

$$S \ |A_{i_1}(\theta_1), \ldots, A_{i_N}(\theta_N)\rangle_{\text{in}} = \sum_{j_1,\ldots,j_N} S^{j_1\cdots j_N}_{i_1\cdots i_N} \ |A_{j_1}(\theta_N), \ldots, A_{j_N}(\theta_1)\rangle_{\text{out}} \tag{1.11}$$

Factorization means that this process can be interpreted as a set of independent and consecutive two-particle scattering processes.

The spacetime picture of this multi-particle factorized scattering is obtained by associating with each particle a line whose slope is the particle's rapidity. The scattering process is thus represented by a planar diagram with N straight world-lines, such that no three ever coincide at the same point. Any world-line will therefore intersect, in general, all the others. The complete scattering amplitude associated with any such diagram is given by the (matrix) product of two-particle S-matrices. For instance, for the four-particle scattering shown in figure 1.2, we obtain

$$S^{j_1 j_2 j_3 j_4}_{i_1 i_2 i_3 i_4}(\theta_1, \theta_2, \theta_3, \theta_4) = \sum_{\substack{k,\ell,m,n,\\ p,q,r,u}} S^{k\ell}_{i_1 i_2}(\theta_{12}) S^{mn}_{\ell i_3}(\theta_{13}) \times S^{pq}_{km}(\theta_{23}) S^{r j_4}_{n i_4}(\theta_{14}) S^{u j_3}_{qr}(\theta_{24}) S^{j_1 j_2}_{pu}(\theta_{34}) \tag{1.12}$$

The kinematical data (the rapidities of all the particles) do not fix a diagram uniquely. In fact, for the same rapidites, we have a whole family of diagrams, differing from each other by the parallel shift of some of the straight world-lines (figure 1.3). The parallel shift of any one line can (and should) be interpreted as a symmetry transformation. It corresponds to the translation of the (asymptotic in- and out-) x co-ordinates of the particle associated to the line. This is indeed

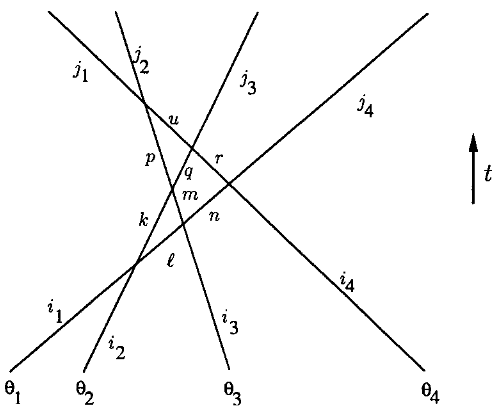

Fig. 1.2. Spacetime diagram of the scattering of four particles. Each line is the world-line of a particle.

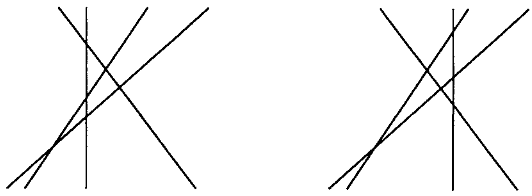

Fig. 1.3 Diagrams differing by the parallel shift of a line.

the symmetry underlying the conservation laws (1.8), which Baxter has called the **Z**-symmetry. Requiring the factorizability condition is equivalent to imposing that the scattering amplitudes of diagrams differing by such parallel shifts should be the same.

For the simple case of three particles, the condition that the factorization be independent of parallel shifts of the world-lines amounts to the following noteworthy factorization equation, which is the necessary and sufficient condition for any two diagrams differing by parallel shifts to have equal associated amplitudes (see figure 1.4):

$$\sum_{p_1,p_2,p_3} S_{i_1 i_2}^{p_1 p_2}(\theta_{12}) S_{p_2 i_3}^{p_3 j_3}(\theta_{13}) S_{p_1 p_3}^{j_1 j_2}(\theta_{23}) = \sum_{p_1,p_2,p_3} S_{i_2 i_3}^{p_2 p_3}(\theta_{23}) S_{i_1 p_2}^{j_1 p_1}(\theta_{13}) S_{p_1 p_3}^{j_2 j_3}(\theta_{12}) \qquad (1.13)$$

Let us stress that the factorization equations (1.13) are a direct consequence of the postulated factorization condition and conservation laws (1.8). Equation (1.13) is the famous Yang–Baxter equation, a matrix equation.

1.1.1 Zamolodchikov algebra

Having sketched the physical meaning of the factorization or Yang–Baxter equation (1.13), we may now turn to a more mathematical interpretation of the factorization conditions from which it derives. To this end, we shall consider a set of operators $\{A_i(\theta)\}$ $(i = 1, \ldots, n)$ associated with each particle i with rapidity

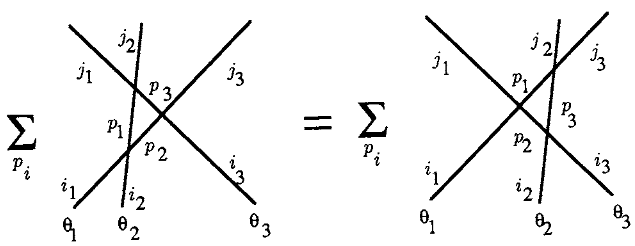

Fig. 1.4 The factorization equations (1.13).

θ, obeying the following commutation relations:

$$A_i(\theta_1)A_j(\theta_2) = \sum_{k,\ell} S_{ij}^{k\ell}(\theta_{12})A_k(\theta_2)A_\ell(\theta_1) \tag{1.14}$$

This equation encodes the two-particle scattering process (1.10), where "collision" has been replaced by "commutation". Furthermore, the relationship between (1.10) and (1.14) becomes evident if we interpret $A_i(\theta)$ as an operator (Zamolodchikov operator) which creates the particle $|A_i(\theta)\rangle$ when it acts on the Hilbert space vacuum $|0\rangle$:

$$A_i(\theta)\,|0\rangle = |A_i(\theta)\rangle \tag{1.15}$$

The factorization equation (1.13) emerges in this context as a "generalized Jacobi identity" of the algebra (1.14), assumed associative. Indeed, let us consider the product of three operators, $A_{i_1}(\theta_1)A_{i_2}(\theta_2)A_{i_3}(\theta_3)$, and let us try to reverse the order of the rapidities. This can be done in two different ways: either

$$\begin{aligned} A_{i_1}(\theta_1)A_{i_2}(\theta_2)A_{i_3}(\theta_3) &\underset{\theta_1\leftrightarrow\theta_2}{=} \sum_{p_1,p_2} S_{i_1 i_2}^{p_1 p_2}(\theta_{12})A_{p_1}(\theta_2)A_{p_2}(\theta_1)A_{i_3}(\theta_3) \\ &\underset{\theta_1\leftrightarrow\theta_3}{=} \sum_{p_1,p_2,p_3,j_3} S_{i_1 i_2}^{p_1 p_2}(\theta_{12})S_{p_2 i_3}^{p_3 j_3}(\theta_{13})A_{p_1}(\theta_2)A_{p_3}(\theta_3)A_{j_3}(\theta_1) \\ &\underset{\theta_2\leftrightarrow\theta_3}{=} \sum_{\substack{p_1,p_2,p_3\\ j_1,j_2,j_3}} S_{i_1 i_2}^{p_1 p_2}(\theta_{12})S_{p_2 i_3}^{p_3 j_3}(\theta_{13})S_{p_1 p_3}^{j_1 j_2}(\theta_{23})A_{j_1}(\theta_3)A_{j_2}(\theta_2)A_{j_3}(\theta_1) \end{aligned} \tag{1.16}$$

or else

$$\begin{aligned} A_{i_1}(\theta_1)A_{i_2}(\theta_2)A_{i_3}(\theta_3) &\underset{\theta_2\leftrightarrow\theta_3}{=} \sum_{p_2,p_3} S_{i_2 i_3}^{p_2 p_3}(\theta_{23})A_{i_1}(\theta_1)A_{p_2}(\theta_3)A_{p_3}(\theta_2) \\ &\underset{\theta_1\leftrightarrow\theta_3}{=} \sum_{p_1,p_2,p_3,j_1} S_{i_2 i_3}^{p_2 p_3}(\theta_{23})S_{i_1 p_2}^{j_1 p_1}(\theta_{13})A_{j_1}(\theta_3)A_{p_1}(\theta_1)A_{p_3}(\theta_2) \\ &\underset{\theta_1\leftrightarrow\theta_2}{=} \sum_{\substack{p_1,p_2,p_3\\ j_1,j_2,j_3}} S_{i_2 i_3}^{p_2 p_3}(\theta_{23})S_{i_1 p_2}^{j_1 p_1}(\theta_{13})S_{p_1 p_3}^{j_2 j_3}(\theta_{12})A_{j_1}(\theta_3)A_{j_2}(\theta_2)A_{j_3}(\theta_1) \end{aligned} \tag{1.17}$$

The equality between these two results is simply the factorization equations (1.13). In practice, it is easier to write down the cubic relations (1.13) with the help of a labeled diagram such as figure 1.4, rather than through the explicit computations (1.16) and (1.17).

Two more conditions are needed to guarantee the consistency of the Zamolodchikov algebra (1.14):

(i) Normalization:

$$\lim_{\theta\to 0} S_{ij}^{k\ell}(\theta) = \delta_i^k \delta_j^\ell \quad \Longleftrightarrow \quad \lim_{\theta\to 0} S(\theta) = \mathbf{1} \tag{1.18}$$

This condition is obtained by setting $\theta_1 = \theta_2$ in (1.14). In physical terms, it means that no scattering takes place if the relative velocity of the two particles vanishes, i.e. if the two world-lines are parallel.

(ii) Unitarity:

$$\sum_{j_1,j_2} S^{i_1 i_2}_{j_1 j_2}(\theta) S^{j_1 j_2}_{k_1 k_2}(-\theta) = \delta^{i_1}_{k_1}\delta^{i_2}_{k_2} \quad \Longleftrightarrow \quad S(\theta)S(-\theta) = \mathbf{1} \tag{1.19}$$

This follows from applying (1.14) twice.

The two conditions above are supplemented, on physical grounds, by two more, namely

(iii) Real analyticity:

$$S^{\dagger}(\theta) = S(-\theta^*) \tag{1.20}$$

which, together with (1.19), implies the physical unitarity condition $S^{\dagger}S = \mathbf{1}$;

(iv) Crossing symmetry:

$$S^{k\ell}_{ij}(\theta) = S^{\bar{\imath}k}_{j\bar{\ell}}(i\pi - \theta) \tag{1.21}$$

where $\bar{\imath}$ and $\bar{\ell}$ denote the antiparticles of i and ℓ, respectively.

1.1.2 Example

Let us consider a theory with only one kind of particle A and its antiparticle $\bar{A}$. According to (1.8), all the possible two-particle scattering processes are shown in figure 1.5.

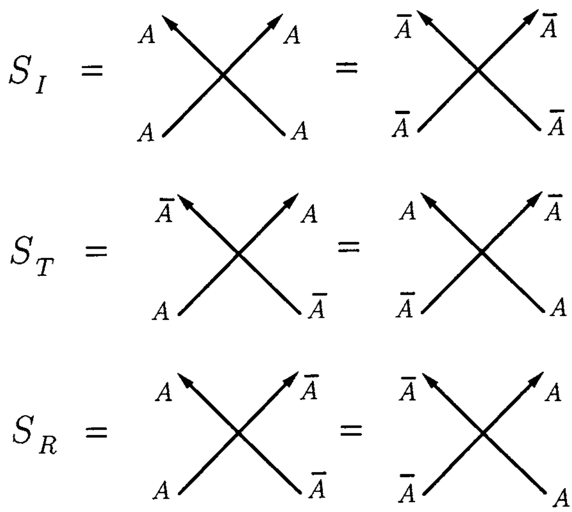

Fig. 1.5 Pair collisions among particles A and antiparticles $\bar{A}$.

Due to CPT invariance, there exist only three different amplitudes (we also assume conservation of particle number, i.e. $\mathbb{Z}_2$ invariance). The scattering amplitude between identical particles (or antiparticles) is denoted S_I, whereas S_T and S_R denote the transmission and reflection amplitudes, respectively. Notice that S_R need not vanish because the masses of a particle and its antiparticle are equal. This illustrates an important consequence of condition (1.8), namely that the redistribution of quantum numbers may take place only among particles with the same mass. In the following, we shall always consider the scattering of n different types of particles (in this example, $n = 2$) with the same mass, whose internal quantum numbers are denoted by the generic label i ($i = 1, \ldots, n$).

The scattering processes of figure 1.5 can be summarized in the following Zamolodchikov algebra:

$$\begin{aligned} A(\theta_1)A(\theta_2) &= S_I(\theta_{12})A(\theta_2)A(\theta_1) \\ A(\theta_1)\overline{A}(\theta_2) &= S_T(\theta_{12})\overline{A}(\theta_2)A(\theta_1) + S_R(\theta_{12})A(\theta_2)\overline{A}(\theta_1) \\ \overline{A}(\theta_1)A(\theta_2) &= S_T(\theta_{12})A(\theta_2)\overline{A}(\theta_1) + S_R(\theta_{12})\overline{A}(\theta_2)A(\theta_1) \\ \overline{A}(\theta_1)\overline{A}(\theta_2) &= S_I(\theta_{12})\overline{A}(\theta_2)\overline{A}(\theta_1) \end{aligned} \tag{1.22}$$

It is not hard to check that the factorization equations for this algebra read as

$$\begin{aligned} S_I S_R' S_I'' &= S_T S_R' S_T'' + S_R S_I' S_R'' \\ S_I S_T' S_R'' &= S_T S_I' S_R'' + S_R S_R' S_T'' \\ S_R S_T' S_I'' &= S_R S_I' S_T'' + S_T S_R' S_R'' \end{aligned} \tag{1.23}$$

where we have set

$$S_a = S_a(\theta_{12}), \qquad S_a' = S_a(\theta_{13}), \qquad S_a'' = S_a(\theta_{23}) \tag{1.24}$$

for $a \in \{I, T, R\}$ to lighten the notation.

The normalization conditions read

$$S_I(0) = 1, \quad S_T(0) = 0, \quad S_R(0) = 1 \tag{1.25}$$

whereas unitarity requires

$$\begin{aligned} S_T(\theta)S_T(-\theta) + S_R(\theta)S_R(-\theta) &= 1 \\ S_T(\theta)S_R(-\theta) + S_R(\theta)S_T(-\theta) &= 0 \end{aligned} \tag{1.26}$$

and the crossing symmetry implies

$$S_I(\theta) = S_T(i\pi - \theta), \qquad S_R(\theta) = S_R(i\pi - \theta) \tag{1.27}$$

In order to find the most general solution to the equations (1.23), we first eliminate S_a'' in favor of S_a and S_a', and thus obtain the compatibility conditions

$$\frac{S_I^2 + S_T^2 - S_R^2}{S_I S_T} = \frac{S_I'^2 + S_T'^2 - S_R'^2}{S_I' S_T'} \tag{1.28}$$

Eliminating S'_a in favor of S_a and S''_a, or S_a in favor of S'_a and S''_a, we find similar equations which imply that the quantity

$$\Delta = \frac{S_I(\theta)^2 + S_T(\theta)^2 - S_R(\theta)^2}{2S_I(\theta)S_T(\theta)} \tag{1.29}$$

is independent of the rapidity θ, and must therefore be related to the coupling constants of the theory.

Clearly, if $\{S_a, S'_a, S''_a\}$ satisfy the Yang–Baxter equation (1.23), then so do $\{\lambda S_a, \lambda S'_a, \lambda S''_a\}$. The overall scale ambiguity can be removed by working with the ratios

$$x(\theta) = \frac{S_I(\theta)}{S_R(\theta)}, \qquad y(\theta) = \frac{S_T(\theta)}{S_R(\theta)} \tag{1.30}$$

whereby equation (1.29) becomes the quadric

$$x^2 + y^2 - 2\Delta xy = 1 \tag{1.31}$$

All three x, y and Δ may be complex. If $|\Delta| \neq 1$, we may parametrize the quadric (1.31) in terms of trigonometric functions of the rapidity θ: the factorizable S-matrix then provides a trigonometric solution to the Yang–Baxter equation. On the other hand, if $|\Delta| = 1$, the S-matrix elements are parametrized by rational functions of θ. The rapidity θ plays the role of the uniformizing parameter for the curve above.

An interesting factorized S-matrix is provided by the sine-Gordon theory, whose lagrangian density is

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{m_\circ^2}{\beta^2}\cos\left(\beta\phi\right) \tag{1.32}$$

If we identify the states A and $\overline{A}$ of (1.22) with the soliton and antisoliton, then the three independent amplitudes are given by

$$\begin{aligned} S_I(\theta) &= \sinh\left[\frac{8\pi}{\eta}(i\pi - \theta)\right] U(\theta) \\ S_T(\theta) &= \sinh\left(\frac{8\pi}{\eta}\theta\right) U(\theta) \\ S_R(\theta) &= i\sin\left(\frac{8\pi^2}{\eta}\right) U(\theta) \end{aligned} \tag{1.33}$$

where $U(\theta)$ is a complicated combination of Γ functions fixed by the normalization and unitarity conditions but not by the Yang–Baxter equation, and η is a renormalization of the coupling constant β of the theory:

$$\eta = \frac{\beta^2}{1 - \frac{\beta^2}{8\pi}} \tag{1.34}$$

For this model, the value of Δ in (1.29) is

$$\Delta = -\cos\left(\frac{8\pi^2}{\eta}\right) \tag{1.35}$$

Recall Coleman's result that for $\beta^2 < 8\pi$ the sine-Gordon hamiltonian is bounded below. The point $\beta^2 = 8\pi$ corresponds to a free fermion model, for which $S_T(\theta) = S_I(\theta) = 1$ and $S_R(\theta) = 0$.

1.2 Bethe's diagonalization of spin chain hamiltonians

Let us start afresh with a different kind of physical system, namely a one-dimensional spin chain. For simplicity, we restrict ourselves for the time being to a periodic one-dimensional regular lattice (a periodic chain) with L sites. At each site, the spin variable may be either up or down, so that the Hilbert space of the spin chain is simply

$$\mathscr{H}^{(L)} = \bigotimes^{L} V^{\frac{1}{2}} \tag{1.36}$$

where $V^{\frac{1}{2}}$ is the spin-$\frac{1}{2}$ irreducible representation of $SU(2)$ with basis $\{|\Uparrow\rangle, |\Downarrow\rangle\}$. By simple combinatorics, the dimension of the Hilbert space is $\dim \mathscr{H}^{(L)} = 2^L$. On $\mathscr{H}^{(L)}$, we consider a very general hamiltonian H, subject to three constraints.

First, we assume that the interaction is of short range, for example only among nearest neighbors, or next to nearest neighbors.

Next, we impose that the hamiltonian H be translationally invariant. Letting e^{iP} denote the operator which shifts the states of the chain by one lattice unit to the right, then this requirement reads as

$$\left[\mathrm{e}^{iP}, H\right] = 0 \tag{1.37}$$

It is more convenient to work with this shift operator directly than with P itself, which is just the lattice version of the momentum operator. From the periodicity of the closed chain, we must have

$$\mathrm{e}^{iPL} = \mathbf{1} \tag{1.38}$$

Finally, we demand that the hamiltonian preserve the third component of the spin:

$$\left[H, S^z_{\text{total}}\right] = \left[H, \sum_{i=1}^{L} S_i^z\right] = 0 \tag{1.39}$$

This requirement allows us to divide the Hilbert space of states into different sectors, each labeled by the third component of the spin or, equivalently, by the total number of spins down. We shall denote by $\mathscr{H}^{(L)}_M$ the subspace of $\mathscr{H}^{(L)}$ with M spins down. Obviously, $\dim \mathscr{H}^{(L)}_M = \binom{L}{M}$, so that $\dim \mathscr{H}^{(L)} = \sum_{M=0}^{L} \dim \mathscr{H}^{(L)}_M$.

We wish to study the eigenstates and spectrum of H. The zeroth sector $\mathscr{H}^{(L)}_0$ contains only one state, the "Bethe reference state" with all spins up. The most natural ansatz for the eigenvectors of H in the other sectors is some superposition of "spin waves" with different velocities. For the first sector, i.e. the subspace of