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Introduction: Some basic concepts of the theory of random processes

Fluctuations in solids are typical random processes. By definition, a random process is a random function $x(t)$ of an independent variable $t$, which, in the theory of fluctuations, is the time. The theory of fluctuations is based on the highly developed mathematical theory of random processes. This chapter is an introduction to the basic concepts and methods of this theory which are used throughout this book. The list of recommended books and reviews is presented at the end of the book.

1.1 Probability density functions; moments; stationary processes

Let $P\{\ldots\}$ be the probability of an event indicated in the curly brackets. The distribution functions of the first, second, and higher order for a given random process $x(t)$ are defined as

$$W_1(x_1, t_1) = P\{x(t_1) \leq x_1\}
$$

$$W_2(x_1, t_1; x_2, t_2) = P\{x(t_1) \leq x_1; x(t_2) \leq x_2\}
$$

$$W_n(x_1, t_1; \ldots; x_n, t_n) = P\{x(t_1) \leq x_1; \ldots; x(t_n) \leq x_n\},
$$

where $x_i$ is the value of the random quantity at instant $t_i$.

For instance, $W_2(x_1, t_1; x_2, t_2)$ is the probability that the random quantity $x$ at instant $t_1$ is less than a given value $x_1$, and at instant $t_2$ is less than $x_2$.

If $W_n(x_1, t_1; \ldots; x_n, t_n)$ are differentiable functions of the variables $x_1, \ldots, x_n$, one may introduce the corresponding probability density functions:

$$w_1(x_1, t_1) = \frac{\partial W_1(x_1, t_1)}{\partial x_1},
$$

$$w_2(x_1, t_1; x_2, t_2) = \frac{\partial^2 W_2(x_1, t_1; x_2, t_2)}{\partial x_1 \partial x_2}.
$$

The functions $w_n$ are called $n$-dimensional probability density functions.
of the random quantity \( x(t) \). They may be defined in a different way. For instance, the two-dimensional probability density function,

\[
w_2(x_1, t_1; x_2, t_2)dx_1dx_2 = P\{x_1 \leq x(t_1) < x_1 + dx_1; x_2 \leq x(t_2) < x_2 + dx_2\}.
\]

(1.1.3)

The probability density functions must be consistent with each other, that is, each \( k \)-dimensional density function can be obtained from any \( n \)-dimensional function with \( n > k \) by integration over ‘extra’ variables:

\[
w_k(x_1, t_1; \ldots; x_k, t_k) = \int dx_{k+1} \ldots dx_n w_n(x_1, t_1; \ldots; x_n, t_n).
\]

(1.1.4)

Random processes are called stationary if all distributions \( w_n(x_1, t_1; \ldots; x_n, t_n) \) remain invariant under any identical shift of all time points \( t_1, \ldots, t_n \). It means that for a stationary process the probability density function \( w_1(x_1, t_1) \) does not depend at all on the time \( t_1 \), the probability density function \( w_2(x_1, t_1; x_2, t_2) \) depends only on the difference \( t_1 - t_2 \), that is, may be written as \( w_2(x_1, x_2, t_1 - t_2) \), and so on.

The mean value of the random variable \( x(t) \) is equal to (for the methods of averaging random quantities see below)

\[
\langle x(t) \rangle = \int_{-\infty}^{+\infty} x(t) dW(x) = \int_{-\infty}^{+\infty} dx x w_1(x, t).
\]

(1.1.5)

The \( r \)-th order central moment is the average value of the random quantity \( (\delta x(t))^r \), where \( \delta x(t) = x(t) - \langle x \rangle \) is the deviation of the random quantity \( x(t) \) from its mean value \( \langle x \rangle \), i.e., the fluctuation:

\[
\langle [\delta x(t)]^r \rangle = \int_{-\infty}^{+\infty} dx (\delta x)^r w_1(x, t).
\]

(1.1.6)

The second-order central moment is the variance, i.e., the mean value of the fluctuation squared.

If the probability density \( w_n(x_1, t_1; \ldots; x_n, t_n) \) is known for different instants \( t_1, \ldots, t_n \), one can calculate the correlation function:

\[
\langle \delta x(t_1) \ldots \delta x(t_n) \rangle = \int dx_1 dx_2 \ldots dx_n \delta x_1 \ldots \delta x_n w_n(x_1, t_1; \ldots; x_n, t_n).
\]

(1.1.7)

The characteristic function of a random quantity is defined as the mean value of the function \( \exp(iux) \), where \( x \) is the random variable and \( u \) is a real parameter:

\[
\phi_x(u) \equiv \langle e^{iux} \rangle = \int_{-\infty}^{+\infty} e^{iux} dW(x).
\]

(1.1.8)

If the distribution function \( W(x) \) is differentiable (see above), the characteristic function is simply the Fourier transform of the probability density
1.1 Probability density functions: moments

function, Eq. (1.1.2):

$$\phi_x(u) = \int_{-\infty}^{+\infty} dx e^{iux} w(x).$$  \hspace{1cm} (1.1.9)

The characteristic function has a very useful property: if the random variable $X$ is a sum of $N$ independent random variables, i.e.,

$$X = \sum_{i=1}^{N} x_i,$$  \hspace{1cm} (1.1.10)

then the characteristic function of $X$ is simply the product of the characteristic functions of the variables $x_i$:

$$\phi_X(u) = \phi_{x_1}(u) \cdot \phi_{x_2}(u) \cdots \phi_{x_N}(u).$$  \hspace{1cm} (1.1.11)

Therefore, in many problems in which the random variable is given by Eq. (1.1.10) it is more convenient to calculate $\phi_X(u)$ first and then the density function $w(X)$ using the inverse Fourier transformation (Eq. (1.1.9)):

$$w(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} du \phi_X(u) e^{-iux}.$$  \hspace{1cm} (1.1.12)

In the theory of fluctuations an important distribution is the normal (Gaussian) distribution. It arises when the random quantity $x(t)$ is a sum of many ($N \gg 1$) independent and identically distributed random quantities. An example is the noise generated in a macroscopic specimen by the random motion of a great number of independent but identical defects, or spins. Let $\xi_1, \ldots, \xi_N$ be independent and identically distributed random quantities and let $x = \xi_1 + \cdots + \xi_N$. If the summands $\xi_1, \ldots, \xi_N$ are small enough and their number $N$ is great enough, that is, if as $N \to \infty$ the mean value of $x(t)$ is equal to $\langle x \rangle$, and the variance of $x(t)$ is equal to $\sigma^2$, the one-dimensional probability density function is

$$w_1(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\delta x)^2}{2\sigma^2}\right],$$  \hspace{1cm} (1.1.13)

where $\delta x = x - \langle x \rangle$ is the fluctuation. Such a distribution is called normal or Gaussian.

A random process is called Gaussian if all its probability density functions are normal for all $n = 1, \ldots$. The $n$-dimensional normal distribution reads:

$$w_n(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det \hat{\lambda}}} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{\lambda}_{ij}^{-1} \delta x_i \delta x_j\right].$$  \hspace{1cm} (1.1.14)

Here $\delta x_i = x_i - \langle x_i \rangle$. The matrix $\hat{\lambda}$ is called the covariance matrix. Its
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elements equal:

\[ \lambda_{ij} = \langle \delta x_i \delta x_j \rangle = \lambda_{ji}. \]  

(1.1.15)

In Eq. (1.1.14) \( \det \hat{\lambda} \) is the determinant of the matrix \( \hat{\lambda} \), \( \lambda_{ij}^{-1} \) are the elements of the matrix \( \hat{\lambda}^{-1} \) which is the reciprocal of \( \hat{\lambda} \).

In particular, the two-dimensional normal distribution equals:

\[ w_2(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho_{12}^2}} \exp\left\{ -\frac{1}{2(1 - \rho_{12}^2)} \left( \frac{(\delta x_1)^2}{\sigma_1^2} + \frac{(\delta x_2)^2}{\sigma_2^2} - 2\rho_{12} \frac{\delta x_1}{\sigma_1} \frac{\delta x_2}{\sigma_2} \right) \right\}. \]  

(1.1.16)

Here \( \sigma_i^2 = \langle (\delta x_i)^2 \rangle \) are the variances, \( \rho_{12} = \langle \delta x_1 \delta x_2 \rangle / \sigma_1 \sigma_2 \) is the correlation coefficient.

Equations (1.1.7) and (1.1.14) imply that for Gaussian random processes all nonzero \( n \)-th order \( (n > 2) \) moments can be expressed in terms of the second-order moments, i.e., the covariances \( \lambda_{ij} \) (pair correlations). In other words, the measurement of higher-order correlations can not give any new information apart from that which is contained in the pair correlation (for the consequences of this property of Gaussian random processes see Sec. 1.2).

Many random processes are similar to electron emission from the cathode of a thermionic tube. The instants of emission of individual electrons are random because the emission events are uncorrelated. Such random processes are called Poissonian. The hops of identical noninteracting defects in a solid are another example of such processes. Let \( n_1 \) be the mean rate of individual events, that is, the mean number of events per unit time. In the case of electron emission, \( n_1 \) is the mean number of electrons emitted per unit time. The mean number of such events during a definite time \( t_m \) is obviously \( \bar{n} = n_1 t_m \). Of course, the actual number \( n \) of events that happen to occur during the time \( t_m \) is a random quantity which fluctuates around its mean value \( \bar{n} \). The problem is to find the probability \( P(n) \) of exactly \( n \) individual events occurring in the time \( t_m = \bar{n}/n_1 \).

Let us divide the time \( t_m \) into \( N \) equal intervals. \( N \) is taken so large or, equivalently, the intervals \( t_m/N \) are taken so small that occurrence of more than one event in one interval is improbable. For each small interval two alternatives are possible: either one individual event or no event may occur. The corresponding probabilities are \( \bar{n}/N \) and \( 1 - \bar{n}/N \), respectively. The probability of each definite sequence of \( n \) ‘full’ and \( N - n \) ‘empty’ intervals (one event or no event occurred, respectively) equals \( (\bar{n}/N)^n(1 - \bar{n}/N)^{N-n} \).
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The total number of different sequences (arrangements) of \( n \) `full' and \( N - n \) `empty' intervals is the number of combinations \( N!/n!(N-n)! \). Thus the probability of \( n \) events occurring equals

\[
\frac{N!}{n!(N-n)!} \left( \frac{\bar{n}}{N} \right)^n \left( 1 - \frac{\bar{n}}{N} \right)^{N-n} = \frac{\bar{n}^n}{n!} \left( 1 - \frac{\bar{n}}{N} \right)^N \cdot \left( 1 - \frac{1}{N} \right) \cdots \left( 1 - \frac{n-1}{N} \right) \left( 1 - \frac{\bar{n}}{N} \right)^{-n}.
\]

The sought probability, which is called the Poisson distribution, follows from the last equation in the limit \( N \to \infty \) at finite \( n \). Since \( \lim_{N \to \infty} (1 - \bar{n}/N)^N = \exp(-\bar{n}) \),

\[
P(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}. \tag{1.1.17}
\]

The statistical characteristics of a random process are nonrandom quantities, which can be, in principle, measured by a corresponding averaging of the random quantity (or quantities). There are two possible procedures of averaging.

1. Averaging over time, i.e., over a sufficiently long record of the random process. In this case, for example, the distribution function \( W_1(x_1) \) of a stationary process is equal to that part of the total time \( t_m \) of measurement of this process during which the random quantity \( x < x_1 \). The mean value of the random quantity is then

\[
\bar{x} = \lim_{t_m \to \infty} \frac{1}{t_m} \int_{-t_m/2}^{t_m/2} dt \ x(t). \tag{1.1.18}
\]

2. Averaging over an ensemble of a large number \( N \) of identical systems, i.e., systems in which identical conditions for the random process and identical means of its measurement are provided. In this case the distribution function, \( W_1(x_1, t_1) \), is equal to the fraction of the total number of ensemble systems in which at instant \( t_1 \) the random quantity \( x < x_1 \). The mean value in this case (denoted by angle brackets) equals:

\[
\langle x \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t). \tag{1.1.19}
\]

Here \( x_i(t) \) is the value of the random quantity at the instant \( t \) in the \( i \)-th system of the ensemble.

One may ask: does a definite mean value exist if one increases the time \( t_m \) of averaging or the number \( N \) of identical systems of the ensemble? This question is fundamental. It is often reduced to the following one: do both averaging procedures (see above) yield identical mean values? If the
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answers are positive, the random process is called ergodic. Ergodicity and nonergodicity are discussed in Sec. 1.4.

1.2 The correlation function

The correlation function is one of the most important characteristics of any random process. Let \( x(t) \) be a quantity that randomly varies in time \( t \) and let \( \langle x \rangle \) be its mean value. The variation in time of the fluctuation, i.e., the deviation of the random quantity from its mean value, \( \delta x(t) = x(t) - \langle x \rangle \), is, of course, also random. The correlation function is a nonrandom characteristic of the kinetics of these random fluctuations: it shows how the fluctuations evolve in time on average.

Consider an ensemble of a large number \( N \) of systems in which identical conditions for the random process under consideration have been created. Let \( \delta x_i(t_1) \) and \( \delta x_i(t_2) \) be the fluctuations in the \( i \)-th system of the ensemble \( (i = 1, \ldots, N) \) at instants \( t_1 \) and \( t_2 \), respectively. The correlation function is defined then by the following equation:

\[
\psi_x(t_1, t_2) \equiv \langle \delta x(t_1) \delta x(t_2) \rangle = \langle x(t_1)x(t_2) \rangle - \langle x(t_1) \rangle \langle x(t_2) \rangle
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \delta x_i(t_1) \delta x_i(t_2).
\]

Here and below the angle brackets denote averaging over an ensemble of identical systems.

In a stationary system the correlation function remains invariable if both instants, \( t_1 \) and \( t_2 \), are shifted identically. In such systems \( \psi_x \) depends only on the difference \( t_1 - t_2 \) (Sec. 1.1).

The correlation function may be determined also by averaging over a sufficiently long record of the random process \( x(t) \) in one system, i.e., by averaging over a long enough time of measurement \( t_m \):

\[
\psi_x(t_1 - t_2) \equiv \delta x(t_1) \delta x(t_2) = \lim_{t_m \to \infty} \frac{1}{t_m} \int_{-t_m/2}^{t_m/2} dt \delta x(t_1 + t) \delta x(t_2 + t).
\]

Both correlation functions (1.2.1) and (1.2.2) coincide if the system is ergodic (see the discussion of ergodicity and nonergodicity of random systems in Sec. 1.4).

According to Eqs. (1.2.1) or (1.2.2), at \( t_1 = t_2 \) the correlation function is simply the variance, i.e., the average fluctuation squared, \( \psi_x(t, t) = \langle (\delta x)^2 \rangle \). As \( |t_1 - t_2| \to \infty \) the correlation function in stable systems falls off to zero. Let the fluctuation in the \( i \)-th system of the ensemble at instant \( t_1 \) be \( \delta x_i(t_1) \). If
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the instant \( t_2 > t_1 \) but is close enough to \( t_1 \), the fluctuation in each system of the ensemble has no time to change its value appreciably, and the fluctuation \( \delta x_i(t_2) \) most probably has the same sign as \( \delta x_i(t_1) \). Therefore, the product \( \delta x_i(t_1)\delta x_i(t_2) > 0 \) in almost all systems of the ensemble. As the instants \( t_1 \) and \( t_2 \) become farther apart, the number of those systems in which the sign of \( \delta x_i(t_2) \) is opposite to the sign of \( \delta x_i(t_1) \) approaches the number of systems in which these signs coincide. Thus, in the sum of Eq. (1.2.1) besides positive terms negative ones also arise. At large enough \( |t_1 - t_2| \) positive and negative summands are equally probable and \( \psi_x(t_1 - t_2) \rightarrow 0 \).

The decay of \( \psi_x(t_1 - t_2) \) to zero as \( |t_1 - t_2| \rightarrow \infty \) may be viewed as if the system ‘forgets’ its ‘initial’ fluctuation. In each physical system there is a characteristic time of ‘forgetting’ the initial fluctuation. This time is usually called the relaxation time. In real systems a number of kinetic processes (they may even be coupled) and, consequently, a number of relaxation times exist. Moreover, in many complex systems a continuous spectrum of such times even exists.

The correlation function can be written in terms of the two-dimensional probability density (Eq. 1.1.7):

\[
\psi_x(t_1, t_2) = \int \int \delta x_1 \delta x_2 w_2(x_1, t_1; x_2, t_2).
\]

(1.2.3)

Here \( \delta x_1 = x_1 - \langle x(t_1) \rangle \), \( \delta x_2 = x_2 - \langle x(t_2) \rangle \).

The two-dimensional probability density may be represented in the form of the product:

\[
w_2(x_1, t_1; x_2, t_2) = w_1(x_1, t_1)P(x_2, t_2|x_1, t_1).
\]

(1.2.4)

Here \( w_1(x, t) \) is the one-dimensional probability density of the random quantity \( x(t) \) at the instant \( t \), \( P(x_2, t_2|x_1, t_1)dx_2 \) is the conditional probability of the quantity \( x \) at time \( t_2 \) being in the interval between \( x_2 \) and \( x_2 + dx_2 \), if in the previous instant \( t_1 \) the value of \( x \) was \( x_1 \). Let us denote by

\[
\langle \delta x(t_2)|\delta x_1, t_1 \rangle = \int \int \delta x_2 P(\delta x_2, t_2|\delta x_1, t_1)
\]

(1.2.5)

the mean value of fluctuation \( \delta x \) at the instant \( t_2 \) under the condition that at a previous instant \( t_1 \) the value of the random quantity was \( x_1 \). In these notations

\[
\psi_x(t_1, t_2) = \int \delta x_1 w_1(\delta x_1, t_1)\langle \delta x(t_2)|\delta x_1, t_1 \rangle.
\]

(1.2.6)

The quantity \( \langle \delta x(t)|\delta x_0, t_0 \rangle \) describes the average evolution (usually, a decay) of the fluctuation, the initial value of which at the instant \( t_0 \) is \( \delta x_0 \). Note that the absolute value of a fluctuation, for a given initial value, may
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either decrease or increase in time. However, on average it usually (but not always) decreases for any given initial value. The correlation function is the value of \( \delta x_1 \langle \delta x(t_2) | \delta x_1, t_1 \rangle \) averaged over the ‘initial’ values \( \delta x_1 \).

Let us select those systems of the ensemble in which the fluctuation at an instant \( t_1 \) is close to a definite value \( \delta x_1 \). After \( t_1 \), eventually the fluctuations in different systems (of the initially selected ones) will be quite different (they are randomized) and their distribution will be the same as that of the fluctuations in the entire ensemble. However, the mean value of the fluctuation for the entire ensemble is zero. Thus, the conditional average value of the fluctuation tends to zero:

\[
\lim_{(t_2-t_1) \to \infty} \langle \delta x(t_2) | \delta x_1, t_1 \rangle = 0.
\]

In the case of a Gaussian random process the conditional mean fluctuation \( \langle \delta x(t) | \delta x_1, 0 \rangle \) and the correlation function are connected by a simple equation. Substitution of \( P(\delta x_2, t_2 | \delta x_1, t_1) = \psi_2(\delta x_1, t_1; \delta x_2, t_2) / \psi_1(\delta x_1, t_1) \) for a Gaussian process (Eq. (1.1.16)) into Eq. (1.2.5) yields after integration over \( \delta x_2 \):

\[
\frac{\langle \delta x(t) | \delta x_0, 0 \rangle}{\delta x_0} = \frac{\psi_x(t)}{\langle \delta x(t) \rangle^2}.
\]  

(1.2.7)

This relationship means that, for a Gaussian process, the time-dependence of the conditional mean fluctuation is the same as that of the correlation function at any initial fluctuation \( \delta x_0 \). If \( \psi_x(t) \) is a monotonically decreasing function of \( |t| \), the fluctuation \( |\delta x(t)| \) at \( t > 0 \) on average decreases at any, arbitrarily small, \( |\delta x(0)| \). Of course, this behavior takes place only on average, the absolute value of a random fluctuation may grow and exceed its value at a preceding instant of time.

Several random quantities \( x_\alpha(t) (\alpha = 1, \ldots, M) \) are often coupled (correlated) with one another. The correlation function is then a matrix:

\[
\psi_{\alpha\beta}(t_1, t_2) = \langle \delta x_\alpha(t_1) \delta x_\beta(t_2) \rangle.
\]  

(1.2.8)

Any function with \( \alpha = \beta \) is called an auto-correlation function, those with \( \alpha \neq \beta \) are the cross-correlation functions of different physical quantities.

The above definition of the correlation function is meaningful only if the quantum-mechanical effects are insignificant and the random quantity \( x(t) \) may be considered as a classical quantity (c-number). In quantum mechanics, a quantum-mechanical operator \( \hat{x} \) is associated with any physical quantity \( x \). Several representations (pictures) of quantum mechanics are known. They are, of course, equivalent. In the Schrödinger representation the wave functions of the system’s states depend on time but the operators do not.
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In the Heisenberg representation, on the contrary, the wave functions (state vectors) are independent of time, and the entire dependence on time $t$ is transferred to the operators $\hat{x}(t)$, which obey the Heisenberg equation of motion:

$$\frac{\partial \hat{x}(t)}{\partial t} = \frac{i}{\hbar}(\hat{H}\hat{x} - \hat{x}\hat{H}).$$

(1.2.9)

Here, $\hat{H}$ is the operator of the system’s energy (Hamiltonian) which is also expressed in terms of Heisenberg operators depending on the same time $t$, $\hbar = h/2\pi$, $h$ is Planck’s constant.

The correlation function $\psi_x(t_1, t_2)$ is more conveniently written in terms of Heisenberg operators of the fluctuating quantities. Obviously, it is expressed in terms of the product of two Heisenberg operators taken at instants $t_1$ and $t_2$, i.e., $\hat{x}(t_1)\hat{x}(t_2)$. However, the correlation function is an observable (measurable) physical quantity, and its values must be real. According to general rules of quantum mechanics, the corresponding operator of the correlation function must be Hermitian. The symmetrical combination,

$$\frac{1}{2}\{\hat{x}(t_1), \hat{x}(t_2)\} = \frac{1}{2}(\hat{x}(t_1)\hat{x}(t_2) + \hat{x}(t_2)\hat{x}(t_1)), \quad (1.2.10)$$

meets this condition (Ekstein & Rostoker, 1955).

As in classical systems, where the correlation function is an average of the product $x(t_1)x(t_2)$, for quantum-mechanical systems it is equal to the expectation (mean value) of the operator given by Eq. (1.2.10). In general, the state of a quantum-mechanical system is determined by a density matrix $\hat{\rho}$, not by a wave function as a system in a ‘pure’ quantum-mechanical state. The expectation of a physical quantity, the operator of which is $\hat{x}$, equals $\langle \hat{x}(t) \rangle = \text{Tr}(\hat{x}(t))$. Here, $\text{Tr}$ denotes the trace (sum of the diagonal matrix elements) of the operator in brackets. In these notations, the correlation function may be written in the form:

$$\psi_x(t_1, t_2) = \frac{1}{2}\langle \{\hat{x}(t_1), \hat{x}(t_2)\} \rangle. \quad (1.2.11)$$

It follows from the very definition of $\psi_x(t_1, t_2)$ that it is a symmetrical function of the instants $t_1$ and $t_2$, i.e.,

$$\psi_x(t_1, t_2) = \psi_x(t_2, t_1). \quad (1.2.12)$$

If the system is a stationary one, the correlation function is a function of the difference $t_1 - t_2$ only (see above). According to Eq. (1.2.12), it is an even function of this difference.

In a more general case when there are several coupled (correlated) quantities $x_\alpha (\alpha = 1, \ldots, M)$, the correlation functions form a matrix (compare with
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Eqs. (1.2.8) and (1.2.11):

$$
\psi_{x\beta}(t_1, t_2) = \frac{1}{2} \langle (\dot{x}_\alpha(t_1)\dot{x}_\beta(t_2) + \dot{x}_\beta(t_2)\dot{x}_\alpha(t_1)) \rangle. \quad (1.2.13)
$$

Its elements obviously satisfy the relation

$$
\psi_{x\beta}(t_1, t_2) = \psi_{\beta x}(t_2, t_1), \quad (1.2.14)
$$

i.e., in a stationary system

$$
\psi_{x\beta}(t_1 - t_2) = \psi_{\beta x}(t_2 - t_1). \quad (1.2.15)
$$

There is another, more profound and less obvious, relationship between the correlation functions $\psi_{x\beta}(t_1, t_2)$. It follows, as does the Onsager’s principle of symmetry of kinetic coefficients (Onsager, 1931), from the symmetry of the equations of motion of any system of particles under the time reversal $t \rightarrow -t$. If the system under consideration is placed in a magnetic field $\mathbf{B}$, this symmetry takes place if simultaneously the magnetic field is also reversed, i.e., $\mathbf{B}$ replaced by $-\mathbf{B}$. This relationship, when applied to stationary systems, takes the form:

$$
\psi_{x\beta}(t_1 - t_2; \mathbf{B}) = \pm \psi_{x\beta}(t_2 - t_1; -\mathbf{B}). \quad (1.2.16)
$$

The plus sign corresponds to the case when the physical quantities $x_\alpha(t)$ and $x_\beta(t)$ are either both invariant under reversal of time $t$, or both change their signs simultaneously. An example of a quantity that changes its sign under time inversion is the velocity of particles. The minus sign corresponds to the case when only one of the two quantities, $x_\alpha$ or $x_\beta$, changes its sign.

Obviously, in the absence of magnetic field ($\mathbf{B} = 0$) Eq. (1.2.16) becomes simpler. The correlation functions then are either strictly even or strictly odd functions of the difference $t_1 - t_2$.

Applying Eq. (1.2.16) to small times $t = t_1 - t_2$, one obtains for $\mathbf{B} = 0$ two relationships for the positive and negative signs in the r.h.s. of Eq. (1.2.16), respectively. The first one is

$$
\dot{\psi}_{x\beta}(0) = 0, \quad (1.2.17)
$$

where $\dot{\psi}$ is the time derivative of $\psi$, while the second one is

$$
\psi_{x\beta}(0) = 0. \quad (1.2.18)
$$

Let $\psi_{x\beta}(\omega)$ be the Fourier transform of the correlation function of a stationary system:

$$
\psi_{x\beta}(\omega) = \int_{-\infty}^{+\infty} d(t_1 - t_2) e^{i\omega(t_1 - t_2)} \psi_{x\beta}(t_1 - t_2). \quad (1.2.19)
$$