1 *Order, Lattices and Domains*

1.1 *Introduction*

DISCUSSION 1.1.1 We shall begin by giving an informal description of some of the topics which appear in Chapter 1. The central concept is that of an ordered set. Roughly, an ordered set is a collection of items some of which are deemed to be greater or smaller than others. We can think of the set of natural numbers as an ordered set, where, *for example*, 5 is greater than 2, 0 is less than 100, 1234 is less than 12687 and so on. We shall see later that one way in which the concept of order arises in computer science is by regarding items of data as ordered according to how much information a certain data item gives us. Very crudely, suppose that we have two programs P and P' which perform identical tasks, but that program P is defined (halts with success) on a greater number of inputs than does P'. Then we could record this observation by saying that P is greater than P'. These ideas will be made clearer in Discussion 1.5.1. We can perform certain operations on ordered sets, for example we have simple operations such as maxima and minima (the maximum of 5 and 2 in the ordered set of natural numbers is 5), as well as more complicated ones such as taking suprema and infima. If the reader has not met the idea of suprema and infima, then he will find the definitions in Discussion 1.2.7. We shall meet examples of ordered sets with given properties; for example, the set of real numbers has the property that the infimum and supremum of any bounded non-empty subset of reals always *exist* (bounded means that every element of the subset is less than a given fixed real and greater than another fixed real). As well as discussing ordered sets in themselves, we shall want to talk about relations between ordered sets and in particular this will include different varieties of function. We will also need to understand the idea that functions *themselves* can be ordered. As an example, consider the function f on the natural numbers which sends n to $n+1$, and g which sends n to $n+4$. Then on every argument, the result of g is greater than that of f, and so we can regard g as greater than f. This completes the informal description of the contents of this chapter. To summarise, Chapter 1 deals with ordered sets, the properties they may have, and relations and functions between ordered sets.

Before beginning in earnest, we shall give a slightly more formal description of the contents of Chapter 1. The account begins with Discussion 1.2.1, which contains a short and terse summary of background material on sets and functions. The idea is simply to fix notation and ideas, and the summary is not a leisurely exposition. We will not introduce every basic mathematical concept that we will be using, but simply give some basic definitions just to give the reader some familiarity with notation and style. For example, while "function" is given a formal definition below (and fixes our notation for functions), it is certainly assumed that the reader has some knowledge of functions, and knows the meaning of injective and surjective function (for which we adopt no special notation). Once the summary is complete, we proceed with discussions of the basic definitions and properties of ordered sets. Different kinds of order are discussed, and concepts such as maximum, greatest element, join and Hasse diagram are defined. We also define the notion of monotone function. With this, we are able to consider some of the most common structures which arise in the theory of ordered sets, such as lattices, Heyting lattices and Boolean lattices. Some basic examples are given, along with some very simple representation theorems which provide information about the way such ordered sets arise. In particular, we describe the idea of a closure system, which gives examples of ordered sets in which the order is given by subset inclusion. Finally we move on to domain theory, once again giving simple examples and proving representation theorems. We also give a number of technical results whose use will only be seen in the later chapters of this book, where domains will provide mathematical models of type theories.

1.2 Ordered Sets

DISCUSSION 1.2.1 We begin with a summary of basic naive set theory. If A and X are sets, we write $A \subseteq X$ to mean A is a subset of X, and $A \subseteq^f X$ to mean that A is a finite subset. A *total function* between a set X and a set Y is a subset $f \subseteq X \times Y$ for which given any $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$. Given $x \in X$ we write $f(x)$ for the unique y such that $(x, y) \in f$. It will often be convenient to write $x \mapsto f(x)$ to indicate that $(x, f(x)) \in f$; for example, if $\mathbb{R}$ is the set of real numbers, then the function f between $\mathbb{R}$ and $\mathbb{R}$, given by $r \mapsto r^2$, is formally the subset

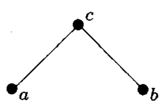
$$\{(r, r^2) \mid r \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}.$$

Often we shall say that f is a function $X \to Y$ and write $f: X \to Y$ in place of $f \subseteq X \times Y$. We shall say (informally) that X and Y are the *source* and *target* of the function f. A function $f: X \to X$ with identical source and target is called an *endofunction* on X. Given functions $f: X \to Y$ and $g: Y \to Z$, we write gf or $g \circ f$ for the function $X \to Z$ defined by $x \mapsto g(f(x))$. A *partial function* between X and Y is a subset $f \subseteq X \times Y$ such that given any elements $(x, y) \in f$ and $(x, y') \in f$ then $y = y'$. If $(x, y) \in f$, we write $f(x)$ for y. If $f: X \to Y$ is a partial function, and given $x \in X$ there is no $y \in Y$ for which $(x, y) \in f$, then we say that f is *undefined* at x, or sometimes simply say that $f(x)$ is undefined. If $f: X \to Y$ is a function and $S \subseteq X$ is a subset of X, then we shall sometimes use the notation $f(S)$ to represent the set $\{f(s) \mid s \in S\}$. If X and Y are any two sets, then the set $X \setminus Y \stackrel{\text{def}}{=} \{x \in X \mid x \notin Y\}$ is the *set difference* of Y from X. If X is a set, then $|X|$ will denote the *cardinality* (size) of X. A *binary relation* R on a set X is any subset $R \subseteq X \times X$. If $x, y \in X$, then we will write xRy for $(x, y) \in R$. R is *reflexive* if for any $x \in X$ we have xRx; *symmetric* if whenever $x, y \in X$ then xRy implies yRx; *transitive* if for any $x, y, z \in X$, whenever we have xRy and yRz then xRz; and *anti-symmetric* if whenever $x, y \in X$, xRy and yRx imply x and y are identical. R is an *equivalence relation* if it is reflexive, symmetric and transitive. Given an equivalence relation R on X, the *equivalence class* of $x \in X$ is the set $[x] \stackrel{\text{def}}{=} \{y \mid y \in X, xRy\}$. We write X/R for the set of equivalence classes $\{[x] \mid x \in X\}$. This completes the summary, and we now move on to the definition of ordered sets.

A *preorder* on a set X is a binary relation $\leq$ on X which is reflexive and transitive. The relation $\leq$ will sometimes be referred to informally as the *order relation* on the set X. It will sometimes be convenient to write $x \geq y$ for $y \leq x$. If at least one of $x \leq y$ and $y \leq x$ holds, then x and y are said to be *comparable*. If neither relation holds, then x and y are *incomparable*. A *preordered* set $(X, \leq)$ is a set equipped with a preorder, that is to say we are given a set (in this case X) along with a preorder $\leq$ on the set X; the set X is sometimes called the *underlying* set of the preorder $(X, \leq)$. Where confusion cannot result, we refer to the preordered set X, or sometimes just the preorder X. The preorder X is said to be *discrete* if any two distinct elements of X are incomparable. If $x \leq y$ and $y \leq x$ then we shall write $x \cong y$ and say that x and y are *isomorphic* elements. Note that we can regard $\cong$ as a relation on X, which is in fact an equivalence relation. If $(X, \leq_X)$ is a preorder, we shall write $S \subseteq X$ to mean that the set S is a subset of the underlying set of X. Of course, we can regard S as a preordered set $(S, \leq_S)$ by restricting the order relation on X to S; more precisely, if $s, s' \in S$, then $s \leq_S s'$ iff $s \leq_X s'$. We shall then say that S has the *restriction order* inherited from X. However, we shall limit the force of the judgement $S \subseteq X$ to mean that S is simply a subset of the underlying set of X. The notation $x \leq S$ will mean that for each $s \in S$, $x \leq s$.

A *partial order* on a set X is a binary relation $\leq$ which is reflexive, transitive and anti-symmetric. A set X equipped with a partial order is called a *partially ordered* set, or sometimes a *poset*. Thus a poset is a preorder which is anti-symmetric. If $x, y \in X$, where X is a poset, then we shall write $x < y$ to mean that $x \leq y$ and $x \neq y$. Given a preorder X then the set of equivalence classes $X/\cong$ can be given a partial ordering by setting $[x] \leq [y]$ iff $x \leq y$ for all $x, y \in X$. The poset $X/\cong$ is called the *poset reflection* of X.

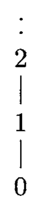
REMARK 1.2.2 We shall use informal pictures, known as *Hasse diagrams*, to describe *partially* ordered sets. *Roughly*, in order to illustrate a finite poset pictorially, we select a distinct point $P(x)$ of the Euclidean plane $\mathbb{R}^2$ for each element x of the poset X and draw a small circle at $P(x)$. If $x < y$ in X and there is no $z \in X$ with $x < z < y$ we draw a line segment $l(x, y)$ joining the circle at $P(x)$ to the circle at $P(y)$, such that the second coordinate of $P(x)$ is strictly less than the second coordinate of $P(y)$. Ensure also that the circle at $P(z)$ does not intersect $l(x, y)$ if z is different from x and y. For example, consider the poset X with underlying set $\{a, b, c\}$ where $a \leq c$, $b \leq c$, $a \leq a$, $b \leq b$ and $c \leq c$. We can draw the Hasse diagram



to represent X. Finally, note that while we can only use this procedure sensibly for finite posets, in practice we shall draw "Hasse diagrams" of infinite posets, making the exact meaning of the picture clear with accompanying mathematics.

EXAMPLES 1.2.3

(1) The set of natural numbers, $\mathbb{N}$, with the usual increasing order is a poset. We will refer to this poset as the *vertical natural numbers*. Although this is an infinite poset, we can draw a diagram to represent it:



(2) See Figure 1.1. Examples (a) and (b) are both finite posets. Example (c) shows that the order in a finite poset can be quite involved. (d) is the poset

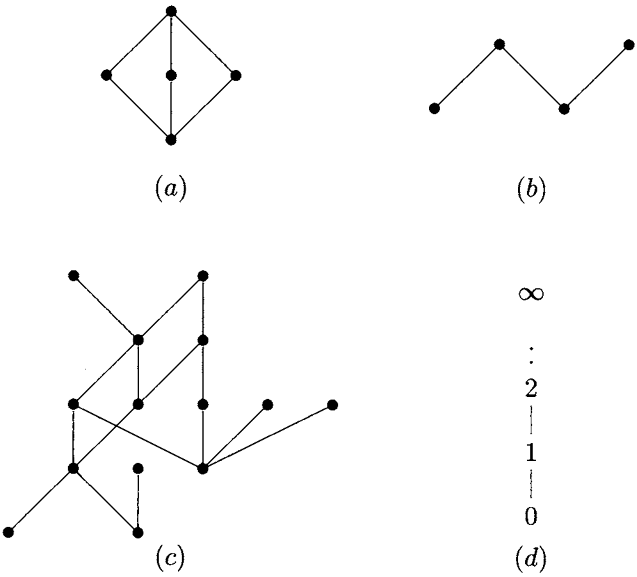


Figure 1.1: Some Examples of Posets.

which is "a copy of the natural numbers (as in example (1)) with a top element added." We will refer to this poset as the *topped* vertical natural numbers. The underlying set of the poset is $\{0, 1, 2, 3, \ldots, \infty\}$.

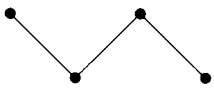
(3) The set $\{A \mid A \subseteq X\}$ of subsets of a set X is often written as $\mathcal{P}(X)$ and is called the *powerset* of X. The powerset is a poset with order given by inclusion of subsets, $A \subseteq B$. The order is certainly anti-symmetric, for if A and A' are subsets of X where $A \subseteq A'$ and $A' \subseteq A$, then $A = A'$. Reflexivity and transitivity are clear.

(4) Given preorders X and Y, their *cartesian product* has underlying set

$$X \times Y \stackrel{\text{def}}{=} \{(x, y) \mid x \in X, y \in Y\}$$

with order given *pointwise*, that is $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$.

(5) If X is a preorder, then X^{op} is the preorder with underlying set X and order given by $x \leq^{op} y$ iff $y \leq x$ where $x, y \in X$. We usually call X^{op} the *opposite* preorder of X. Of course any poset is certainly also a preorder. The following Hasse diagram is a picture of the opposite of the poset (b):



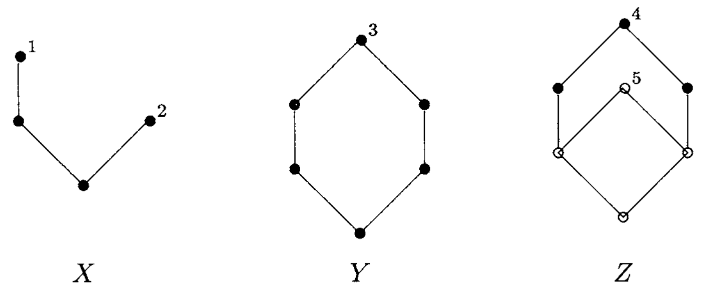


Figure 1.2: Illustrating some Definitions.

DISCUSSION 1.2.4 We now give some more definitions. Suppose that X is a preorder and A is a subset of X. An element $x \in X$ is an *upper bound* for A if for every $a \in A$ we have $a \leq x$ (or we can just write $A \leq x$, using the informal notation given in Discussion 1.2.1). An element $x \in X$ is a *lower bound* for A if $x \leq A$. An element $x \in X$ is a *greatest element* of A if it is an upper bound of A which belongs to A; x is a *least element* of A if it is a lower bound of A and belongs to A. An element $a \in A$ is *maximal* if for every $b \in A$ we have $a \leq b$ implies that $a \cong b$. An element $a \in A$ is *minimal* if for every $b \in A$ we have $b \leq a$ implies that $b \cong a$. We can prove a useful little result about greatest and least elements:

PROPOSITION 1.2.5 Let X be a preordered set and A a subset of X. Then greatest and least elements of A are unique up to isomorphism if they exist.

PROOF Let a and a' be greatest elements of A. By definition, a is an upper bound of A, and also $a' \in A$. Hence $a' \leq a$. Similarly $a \leq a'$. Hence $a \cong a'$. The proof for least elements is essentially the same. □

EXAMPLES 1.2.6

(1) Consider the posets illustrated in Figure 1.2. Of course, any poset is certainly a preorder, and we consider examples of the above definitions. In poset X the elements 1 and 2 are maximal. In poset Y, the element 3 is a maximal element which is the greatest element of Y. In poset Z, element 4 is greatest and maximal in Z and 5 is maximal in the subset of Z indicated by the light circles.

(2) Consider the poset of natural numbers $\mathbb{N}$ with its usual increasing order and the subset $S \stackrel{\text{def}}{=} \{25, 65, 100\}$. Examples of lower bounds for S are 0, 10 and 25, and examples of upper bounds are 100, 105, 1253 and 245.

DISCUSSION 1.2.7 The notions of upper bound, maximal element and so on give us mathematical tools for the description of the structure of preordered sets. The reader is probably familiar with the everyday notions of maximum and minimum, and our definitions of greatest and least elements correspond to such notions. Unfortunately, such ideas are not quite general enough for our purposes. We shall now define the concept of meet and join which is a generalisation of the notion of maximum and minimum.

Let X be a preordered set and $A \subseteq X$. A *join* of A, if such exists, is a least element in the set of upper bounds for A. A join is sometimes called the *least upper bound* or a *supremum*. A *meet* of A, if it exists, is a greatest element in the set of lower bounds for A. A meet is sometimes called the *greatest lower bound* or *infimum*. Note that meets and joins are defined as greatest and least elements; so from Proposition 1.2.5 we know that meets and joins are determined up to isomorphism if they exist. If the subset A has at least one join, then we will write $\bigvee A$ for a choice of one of the joins of A. Similarly, if the subset A has at least one meet, then we will write $\bigwedge A$ for a choice of one of the meets of A. If we wish to draw attention to the ordered set with respect to which a join and meet are being taken (in this case X) we shall write $\bigvee_X A$ and $\bigwedge_X A$ respectively. Note that the join is characterised by the property that for every $x \in X$ we have $\bigvee A \leq x$ iff $A \leq x$; this amounts to a formal statement that a join is by definition a least element in a set of upper bounds. Using the notation described on page xvi, we could also say that $\bigvee A$ is a join for the subset $A \subseteq X$ if for every $x \in X$ we have

$$\frac{\bigvee A \leq x}{\overline{A \leq x}}$$

Similarly, $\bigwedge A$ is a meet of a subset A of X if for every $x \in X$ we have

$$\frac{x \leq \bigwedge A}{\overline{x \leq A}}$$

Some special points deserve attention.

- Let X be a non-empty discrete preorder X, and $A \subseteq X$ a non-empty subset. Then A only has a meet or join if A is a singleton set. Clearly, for any $x \in X$, we have $\bigwedge\{x\} = x$ and $\bigvee\{x\} = x$.

• Consider the empty set, $\varnothing \subseteq X$. Then $\bigvee \varnothing$, if such exists, is written $\bot$ and is called a *bottom* of X. Note that a bottom element satisfies the property that for any $x \in X$ we have $\bot \leq x$. Similarly, $\bigwedge \varnothing$, if such exists, is written $\top$ and is called the *top* of X; it satisfies $x \in X$ implies $x \leq \top$.

• Consider a two element subset $\{a, b\} \subseteq X$. Write $a \vee b$ for $\bigvee\{a, b\}$ and call this a (binary) join of a and b. Similarly $a \wedge b$ is a (binary) meet of a and b. If we unravel the definitions, it can be seen that binary joins are characterised by the property that for every $x \in X$ we have $a \vee b \leq x$ iff $a \leq x$ and $b \leq x$; and binary meets by asking that for any $x \in X$ we have $x \leq a \wedge b$ iff $x \leq a$ and $x \leq b$.

EXERCISE 1.2.8 Make sure you understand the definition of meet and join in a preorder X. Think of some simple finite preordered sets in which meets and joins do not exist. Now suppose that X is a poset (and thus also a preorder). Show that meets and joins in a poset are unique if they exist.

DISCUSSION 1.2.9 A subset C of a preorder X is called a *chain* if for every $x, y \in C$ we have $x \leq y$ or $y \leq x$. We shall often simply refer to a chain in X. C is called an *ω-chain* if its elements can be indexed by the natural numbers, say $C \stackrel{\text{def}}{=} \{x_n \mid n \in \mathbb{N}\}$. C is an *anti-chain* if for every $x, y \in C$ then $x \leq y$ iff $x \geq y$. A subset D of X is called *directed* if every finite subset of D has an upper bound in D. Note that we regard the empty set as finite; thus any directed subset is non-empty by definition. We say the poset X is *directed* if any finite subset of X has an upper bound in X. A subset I of a preorder X is *inductive* if given a directed subset $D \subseteq X$ for which $D \subseteq I$ then $\bigvee_X D \in I$. We shall say that a preorder X is a *chain* or *anti-chain* if the underlying set X is such. Given a subset A of a preorder X, then the *up-set* of A is defined to be $A{\uparrow} \stackrel{\text{def}}{=} \{x \in X \mid x \geq A\}$ and the *down-set* is $A{\downarrow} \stackrel{\text{def}}{=} \{x \in X \mid x \leq A\}$. So the up-set of A is the set of all upper bounds of A, and the down-set of A is the set of all lower bounds of A. We shall write $x{\downarrow}$ for $\{x\}{\downarrow}$ and $x{\uparrow}$ for $\{x\}{\uparrow}$, where $x \in X$.

REMARK 1.2.10 This example shows why we take care with definitions involving subsets of preorders and posets. Let $X \stackrel{\text{def}}{=} \{1, 2, \ldots, n, n{+}1, \ldots, \infty, \top\}$ be the poset with partial order "generated" by

$$1 \leq 2 \leq 3 \leq 4 \leq 5 \ldots \leq \infty \leq \top.$$

Let $I \stackrel{\text{def}}{=} X \setminus \{\infty\}$ and let $D \stackrel{\text{def}}{=} X \setminus \{\infty, \top\}$, and refer to Figure 1.3. If we preorder I with the restriction order from X (so that I is a copy of the topped

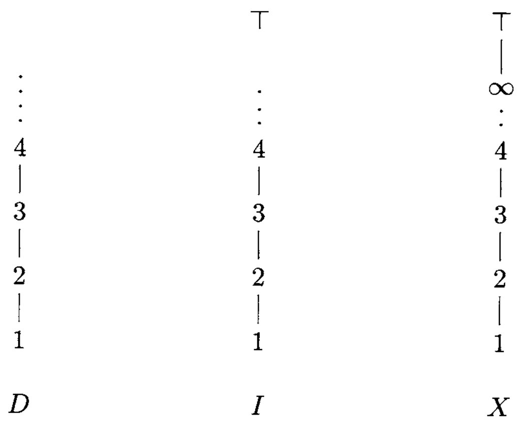


Figure 1.3: A Subset of a Poset which is not Inductive.

vertical natural numbers) then $\bigvee_I D$ exists in the preorder I and is $\top$ with respect to this order; *but* I is not an inductive subset of X because $\bigvee_X D = \infty$ which is not an element of I. It is sometimes tempting for beginners to glance at a *subset* such as I and think it must be inductive with the restriction order. See also Example 1.3.3.

EXERCISE 1.2.11 Let C and C' be chains. Show that the set of pairs (c, c'), where $c \in C$ and $c' \in C'$, is also a chain when ordered lexicographically. Show that the set of pairs with the pointwise order is a chain just in case at most one of C or C' has more than one element.

DISCUSSION 1.2.12 The existence of meets and joins for certain kinds of subsets of preordered sets is known as completeness and cocompleteness respectively. If P is a property of a subset A of the preorder X, and meets exist for all such subsets A, then we say that X is *P-complete*; dually X is *P-cocomplete* if joins exist for subsets A with property P. For example, suppose that X has binary meets and a top element. Then by induction it is easy to see that X has meets of all finite subsets, and we say that X is *finitely* complete. If X has joins of all directed subsets then it is said to be *directed* cocomplete, and if X has joins of ω-chains it is said to be ω-cocomplete. If X has meets or joins of *all* subsets then it is said to be *complete* or *cocomplete* respectively. We can give a very useful result which states that a preorder X is complete if and only if it is cocomplete:

LEMMA 1.2.13 A preorder X has all meets just in case it has all joins.

PROOF Suppose that A is any subset of X. Note that one has

$$\bigwedge A \stackrel{\text{def}}{=} \bigvee\{x \mid x \in X \text{ and } x \leq A\} \quad \text{and} \quad \bigvee A \stackrel{\text{def}}{=} \bigwedge\{x \mid x \in X \text{ and } A \leq x\}.$$

□

EXAMPLES 1.2.14

(1) Given a set X, the powerset poset $\mathcal{P}(X)$ is both complete and cocomplete. Meets are given by intersections and joins by unions. The top element is of course X and the bottom element $\varnothing$.

(2) Suppose that a preorder X is finitely complete and cocomplete, that is to say X has meets and joins of all finite subsets. We regard the empty subset as being finite and thus X has top and bottom elements.

DISCUSSION 1.2.15 We now turn our attention to notions of relations between preordered sets, and in particular to functional relations. If we talk of a function between the preordered sets X and Y we shall simply mean that we are given a function between the underlying sets. Such a function is said to be *monotone* if for $x, y \in X$ we have $x \leq y$ implies $f(x) \leq f(y)$; and *antitone* if $x \leq y$ implies $f(y) \leq f(x)$. We often refer to such a monotone function as a *homomorphism of preorders*. Roughly one thinks of a homomorphism as a function which preserves structure; in the case of a preorder, this structure is just the order relation. A monotone function may alternatively be called an *order preserving* function. f is said to *reflect order* if given any $x, y \in X$, $f(x) \leq f(y)$ implies $x \leq y$. The posets X and Y are *isomorphic* if there are monotone functions $f\colon X \to Y$ and $g\colon Y \to X$ for which $gf = id_X$ and $fg = id_Y$. The monotone function g is an *inverse* for f; and likewise f is an inverse for g. We say that f is an *isomorphism* if such an inverse g exists. The set $X \Rightarrow Y$ is defined to have elements the monotone functions with source X and target Y, that is functions $X \to Y$. This set can be regarded as a preorder by defining a relation $f \leq g$ iff given any $x \in X$ we have $f(x) \leq g(x)$, where $f, g\colon X \to Y$. This ordering is often referred to as the *pointwise* order. We have the following proposition:

PROPOSITION 1.2.16 The identity function on any preordered set is monotone, and the composition of two monotone functions is another monotone function. Now let X, Y and Z be preordered sets. The composition function

$$\circ\colon (Y \Rightarrow Z) \times (X \Rightarrow Y) \to (X \Rightarrow Z)$$