Cambridge University Press
978-0-521-45689-0 - An Introduction to Twistor Theory, Second Edition
S. A. Huggett and K. P. Tod
Excerpt
More information

Chapter 1

Introduction

Twistor Theory began as a subject in the late 1960's with the appearance of Penrose's two papers (1967, 1968a). A more definitive statement of its aims and accomplishments was 'Twistor Theory: An Approach to the Quantisation of Fields and Space-time' (Penrose and MacCallum), which appeared in 1973.

What is apparent from the start is the breadth of application which Roger Penrose saw for it. In the decades since then the subject has grown in different direction to the extent that different traditions have emerged.

There is a purely mathematical strain which refers to the Penrose Transform and is interested in its geometrical and complex analytic features, frequently in a positive definitve setting.

There is a quantum field theoretic strain concerned with elementary particles and their interactions in Minkowski space.

There is a modest point of view which simply holds that the theory is useful for solving some non-linear equations and the object is to discover which ones, and there is a full-blooded strain which holds that the repeated occurence and usefulness of complex numbers and complex analyticity tells one something fundamental about the physical world.

The full-blooded strain of twistor philosophy may be seen in for example (Penrose 1975). One place where the diversity of twistor theory is manifest is the Twistor Newsletter, an informal publication produced by the Oxford group about twice a year. The content of the first ten Newsletters was published as 'Advances in Twistor Theory' (Hughston and Ward 1979) and later articles were collected in 'Further Advances in Twistor Theory' (Mason and Hughston 1990).

In this book our aim is simply to give an introduction to the subject and point the reader in the direction of these other possibilities. One thing we do want to emphasize is that a large part of the material falls under the heading of space-time geometry and indeed a course on that subject could be made out of the first six chapters.

The plan of the book is as follows.

We begin in chapter 2 with a review of tensor algebra and calculus as a reminder and to fix conventions. Lorentzian spinors at a point are introduced in chapter 3 and spinor algebra is developed. Also the definition of a complex manifold is given with projective spinors as a paradigm. In chapter 4 we define spinor bundles and the spinor covariant derivative and introduce various spinor differential equations, notably the zero rest mass free field equations and the twistor equation.

In attempting to define a Lie derivative of spinors, we are led in chapter 5 to a consideration of compactified Minkowski space and conformal invariance.

The geometry of null geodesic congruences is discussed in terms of spinors in chapter 6 and the connection between shear and complex analyticity is noticed. We find shear-free congruences in terms of free analytic functions of three variables.

Twistors are introduced in chapter 7, first as spinor fields solving the twistor equation and acted on by the conformal group. The geometrical correspondence between twistor space and complexified compactified Minkowski space is developed and other pictures of a twistor, as an α-plane or a Robinson congruence are given.

In chapter 8 we give the twistor contour integral solution of the zero rest mass free field equations and in attempting to understand curious features of the solution we are led in chapter 9 to sheaf cohomology. After an informal account of sheaf cohomology we return in chapter 10 to the zero rest mass equations and interpret the contour integrals cohomologically.

In chapters 11 and 12 we describe two 'active' constructions where field equations in space-time are coded into deformations of complex structure in corresponding twistor spaces. These are the construction due to Ward for solving the self-dual Yang–Mills equations and the construction due to Penrose for solving the self-dual Einstein equations.

An application of twistor theory in conventional general relativity is discussed in chapter 13. This is Penrose's proposal for a quasi-local momentum-angular-momentum in an arbitrary curved space-time.

The sheaf cohomology of chapter 9 is extended in chapter 14, in which we show how to describe some spaces of multilinear functionals of zero rest mass fields.

Finally in chapter 15 we briefly mention some other developments in twistor theory not mentioned elsewhere!

We have provided exercises throughout, both within the text and grouped at the end of the chapters and have included a chapter (16) of hints, solutions and notes. Some of the exercises are just problems on the material covered but others are open-ended and intend to lead the reader into regions which we don't have the space to cover more fully. (Some of *these* exercises are then referred to later in the text!)

Chapter 2

Review of Tensor Algebra and Calculus

This chapter is chiefly intended as a reminder and to fix conventions (which largely follow Penrose and Rindler 1984).

We shall be concerned with a real four-dimensional vector space V, its dual $V^*$and its complexification $V_{\mathbf{C}} = V \otimes \mathbf{C}$.

Vectors or elements of V are written V^a; covectors, elements of V^*, are written W_a and the pairing

$$V \times V^* \to \mathbf{R} \text{ is } (V^a, W_a) \to V^a W_a.$$

We follow the abstract index convention of Penrose (1968b) in which the index a on V^a is simply an indication that the object is a vector, rather than one of a set of numbers.

Higher valence tensors are elements of tensor products of V with V^* as, e.g. :

$$P^{a_1 \dots a_r}{}_{b_1 \dots b_s} \in \underbrace{V \otimes \dots \otimes V}_{r} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{s}.$$

Algebraic operations on tensors which we shall require are:

i) Contraction: $P^{ab \dots c}{}_{de \dots f} \to P^{ab \dots c}{}_{ae \dots f}$

ii) Symmetrisation: $P^{(a \dots b)} = \frac{1}{r!} \sum_\sigma P^{\sigma(a) \dots \sigma(b)}$

where $P^{a \dots b}$ has r indices and the sum is over all permutations. For example

$$P^{(ab)} = \frac{1}{2}(P^{ab} + P^{ba}).$$

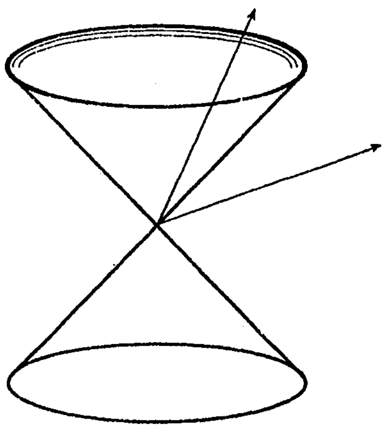

Figure 2.1. The null cone in V separating time-like from space-like.

iii) Skew or anti-symmetrisation: $P^{[a\ldots b]} = \frac{1}{r!}\sum_{\sigma}(\text{sign}\sigma)P^{\sigma(a)\ldots\sigma(b)}$

where sign σ is ± 1 according as σ is an even or odd permutation. For example

$$P^{[ab]} = \frac{1}{2}(P^{ab} - P^{ba}).$$

A skew-symmetric tensor of valence 2 will be referred to as a *bivector*.

We shall suppose that V comes equipped with a Lorentzian metric, i.e. a symmetric non-degenerate tensor η_{ab} which is equal to diag $(1, -1, -1, -1)$ in an orthonormal frame.

Non-zero vectors in V are characterized as time-like, space-like or null according as their 'length' $\eta_{ab}V^aV^b$ is positive, negative or zero. This gives rise to the characteristic picture of the null cone in V (see figure 2.1).

Removing the origin O disconnects the null cone, so that time-like and null vectors can be further distinguished into two classes, one of which can be labelled future-pointing and the other past-pointing.

The metric allows the identification of V and V^*:

$$V \to V^*\ ;\ V^a \to V_a = \eta_{ab}V^b$$

together with the inverse:

$$V^* \to V\ ;\ V_b \to V^b = \eta^{bc}V_c$$

where $\eta^{ab}\eta_{bc} = \delta^a{}_c$.

Usually a choice of orientation for V is made, that is, a choice of skew tensor $\epsilon_{abcd} = \epsilon_{[abcd]}$ with

$$\epsilon_{abcd}\epsilon^{abcd} = -24.$$

Then a right-handed orthonormal frame (T^a, X^b, Y^c, Z^d) is one with

$$\epsilon_{abcd}T^aX^bY^cZ^d = 1.$$

With the aid of ϵ_{abcd} and the metric we may define the dual, ${}^*F_{ab}$, of a bivector F_{ab} as:

$${}^*F_{ab} = -\frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}.$$

As a consequence of the signature of the metric, we find

$${}^{**}F_{ab} = -F_{ab}.$$

Thus the eigenvalues of duality on bivectors are $\pm i$ and therefore the eigenbivectors are necessarily complex.

The *Lorentz group* $L = O(1,3)$ is the group of endomorphisms of V preserving η_{ab}:

$$\Lambda^a{}_b \in L \Leftrightarrow \Lambda^a{}_b\Lambda^c{}_d\eta_{ac} = \eta_{bd}$$

or, in matrix notation,

$$\eta = \Lambda^T\eta\Lambda \text{ where } \eta = \text{diag}(1,-1,-1,-1).$$

Clearly $\det\Lambda = \pm 1$. Lorentz transformations with negative determinant change the orientation since

$$\epsilon_{abcd}\Lambda^a{}_p\Lambda^b{}_q\Lambda^c{}_r\Lambda^d{}_s = (\det\Lambda)\epsilon_{pqrs}.$$

Also a Lorentz transformation will interchange the future and past null cones if $\Lambda^0{}_0 < 0$. Thus L has four components which may be represented as

$$L = L_+^\uparrow \cup L_+^\downarrow \cup L_-^\uparrow \cup L_-^\downarrow$$

where $\pm$ indicates the sign of the determinant and $\uparrow$ indicates $\Lambda^0{}_0 > 0$. The first component $L_+^\uparrow$ contains the identity and is referred to as the *proper orthochronous Lorentz group*.

As examples of Lorentz transformations, we have

i) $\text{diag}(-1,1,1,1) \in L_-^\downarrow$: time-reflection

ii) $\text{diag}(1,-1,1,1) \in L_-^\uparrow$: space-reflection

iii) $\begin{bmatrix} \cosh\psi & 0 & 0 & \sinh\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\psi & 0 & 0 & \cosh\psi \end{bmatrix} \in L_+^\uparrow$: 'boost' in the (03) plane.

The way to calculate the dimension of the Lorentz group is to consider infinitesimal Lorentz transformations: in matrix form

$$\Lambda = I + \epsilon S.$$

Then, to first order in ϵ,

$$\Lambda^T \eta \Lambda = \eta \Rightarrow S^T \eta + \eta S = 0.$$

This gives ten conditions on the sixteen components of S, so that L is six-dimensional.

We assume the reader is familiar with the basic machinery of differential geometry, i.e. the definitions of a smooth manifold and its tangent and cotangent bundles as given for example in Hicks (1965) or Do Carmo (1976, 1992).

We are thinking of the vector space V above as being the tangent space $T_pM = (TM)_p$ at a point p of a real four-dimensional manifold M. The metric on V comes from a metric on M, that is a smooth, valence 2, non-degenerate symmetric tensor field g_{ab} with signature -2. This allows the definition of the orthonormal frame bundle B of M and also determines a unique, torsion-free, metric-preserving connection, the Levi-Civita connection, which we denote ∇_a.

Commuted derivatives give rise to curvature via the Ricci identity:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)V^d = R_{abc}{}^d V^c \tag{2.1}$$

which defines the Riemann tensor $R_{abc}{}^d$. The Ricci tensor R_{ab} and Ricci scalar R are defined by

$$R_{ab} = R_{acb}{}^c\ ;\ R = g^{ab}R_{ab},$$

and the Einstein field equations of general relativity are

$$R_{ab} - \frac{1}{2}Rg_{ab} = -T_{ab}$$

with a suitable choice of units, where T_{ab} is the stress–energy tensor of matter. The Riemann tensor satisfies the *first Bianchi identity*:

$$R_{a[bcd]} = 0 \tag{2.2}$$

which implies the interchange symmetry of the Riemann tensor and the symmetry of the Ricci tensor:

$$R_{abcd} = R_{cdab}\ ;\ R_{ab} = R_{ba},$$

and also the *second Bianchi identity* (which is usually referred to as *the* Bianchi identity):

$$\nabla_{[a} R_{bc]de} = 0. \tag{2.3}$$

Finally, the *Lie derivative* along a vector field X^a is defined by

i) $\mathcal{L}_X f = X^a \nabla_a f$ for functions f

ii) $\mathcal{L}_X Y^a = X^b \nabla_b Y^a - Y^b \nabla_b X^a$

iii) for arbitrary valence tensors, extend by the Leibniz rule.

The Lie derivative is actually part of the differential structure of M and is defined prior to the assumption of a metric. Put another way, the above definition is independent of (symmetric) connection.

Our next step is to introduce spinors and develop the algebra and calculus of spinors by analogy with the tensor algebra and calculus.

Exercises 2

a) Given a tensor R_{abcd} with the symmetries of the Riemann tensor:

$$R_{abcd} = R_{[ab]cd} = R_{ab[cd]}\ ;\ R_{a[bcd]} = 0$$

define $S_{abcd} = R_{c(ab)d}$.

Show that S_{abcd} has the symmetries of R_{abcd} but with skew-symmetrisers replaced by symmetrisers (i.e. square brackets replaced by round brackets). Find an expression for R_{abcd} in terms of S_{abcd}.

b) A bivector F_{ab} is said to be simple iff it can be written as the skew outer product of two vectors $2U_{[a}V_{b]}$. Show that F_{ab} simple $\Leftrightarrow \epsilon^{abcd} F_{ab} F_{cd} = 0$
$\Leftrightarrow {}^*F^{ab} F_{ab} = 0$.

c) Show that $\mathcal{L}_X g_{ab} = 2\nabla_{(a} X_{b)}$, where ∇_a is the Levi-Civita connection preserving the metric g_{ab}.

d) Show that $\mathcal{L}_X {}^*F_{ab} = {}^*(\mathcal{L}_X F_{ab})$ iff $\mathcal{L}_X g_{ab} = \lambda g_{ab}$ for some function λ. A vector field X^a with this property is known as a conformal Killing vector (see chapter 5).

e) If X^a, Y^a are two conformal Killing vectors show that $\mathcal{L}_X Y^a$ is also.

Chapter 3

Lorentzian Spinors at a Point

Given a vector $V^a \in V$ with components (V^0, V^1, V^2, V^3) in some orthonormal frame, we may define a Hermitian matrix $\Psi(V^a)$ by (Penrose 1974; Penrose and Rindler 1984; Pirani 1965)

$$\Psi(V^a) = V^{AA'} = \begin{bmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{bmatrix}. \quad (3.1)$$

The indices A, A' have the ranges $0, 1$ and $0', 1'$ respectively. The significance of the prime will become apparent!

Clearly (3.1) gives a one-one correspondence between 2×2 Hermitian matrices and elements of V. Further, the determinant of the matrix is half the length of the vector:

$$\det \Psi(V^a) = \frac{1}{2}\eta_{ab} V^a V^b.$$

If we multiply the matrix $\Psi(V^a)$ on the left by an element of $SL(2, \mathbb{C})$ (i.e. a 2×2 matrix with complex entries and unit determinant) and on the right by its Hermitian conjugate:

$$V^{AA'} \to \tilde{V}^{AA'} = t^A{}_B V^{BB'} \bar{t}^{A'}{}_{B'}$$

where $\begin{bmatrix} t^0{}_0 & t^0{}_1 \\ t^1{}_0 & t^1{}_1 \end{bmatrix} \in SL(2, \mathbb{C})$, and

$$\bar{t}^{A'}{}_{B'} = \overline{t^A{}_B},$$

then the result will be another Hermitian matrix and the determinant will be unchanged. This process therefore defines a linear transformation on the vector V^a preserving its length, i.e. a Lorentz transformation:

$$V^a \to \tilde{V}^a = \Lambda^a{}_b V^b.$$

Thus we have a map $SL(2,\mathbb{C}) \to L$. The following properties of this map may be readily established:

i) it is a group homomorphism;

ii) it is into $L_+^\uparrow$;

iii) the kernel consists of $\pm I$ in $SL(2,\mathbb{C})$, where I is the identity matrix.

Since $SL(2,\mathbb{C})$ is also a six-parameter group the map is necessarily onto and so is a 2-1 isomorphism. In exercise 3a we establish that $SL(2,\mathbb{C})$ is simply connected, so that this map exhibits $SL(2,\mathbb{C})$ as the universal cover of $L_+^\uparrow$.

The Lorentz transformations in $L_+^\uparrow$ leaving invariant the time-like vector in the chosen orthonormal tetrad evidently define a three-dimensional rotation group, $SO(3)$. The matrix Ψ_0 corresponding to the time-like vector is proportional to the unit matrix, so that the $SO(3)$ subgroup of $L_+^\uparrow$ is covered by an $SU(2)$ subgroup of $SL(2,\mathbb{C})$.

As an example of the map Ψ, the matrix

$$t = \begin{pmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{pmatrix}$$

determines a boost in the (03) plane while

$$t = \begin{pmatrix} e^{\frac{i\psi}{2}} & 0 \\ 0 & e^{-\frac{i\psi}{2}} \end{pmatrix}$$

determines a rotation through ψ in the (12) plane.

[A topological aside: a path from I to $-I$ in $SL(2,\mathbb{C})$ will correspond in $L_+^\uparrow$ to a path beginning and ending at I, but which cannot be shrunk to a point. For example

$$t = \begin{pmatrix} e^{\frac{i\lambda}{2}} & 0 \\ 0 & e^{\frac{i\lambda}{2}} \end{pmatrix} \quad 0 \le \lambda \le 2\pi$$