1 Objects and Logic

§1. Abstract objects

The language of mathematics speaks of objects. This is a rather trivial statement; it is not certain that we can conceive any developed language that does not. What is of interest is that, taken at face value, mathematical language speaks of objects distinctively mathematical in character: numbers, functions, sets, geometric figures, and the like. To begin with, they are distinctive in being abstract.

Roughly speaking, an object is abstract if it is not located in space and time and does not stand in causal relations. This criterion gives rise to some uncertain cases and would not be accepted by all philosophers.1

1 This is close to what David Lewis calls the Way of Negation in his discussion of the abstract-concrete distinction in On the Plurality of Worlds, §1.7. Criticisms of the criterion are to be found there and in Burgess and Rosen, A Subject with no Object, §I.A.1.b. The counterexamples offered are for the most part not mathematical objects. However, these authors claim that some abstract objects of the kind called quasi-concrete in §7 below are located. There is one case relevant to mathematics, that of sets of concrete objects, which will be discussed there, in note 57.

A reservation about the causal aspect that is worth mentioning is the following. There has been some dispute about the kind of entities that enter into causal relations connected with the dispute about whether events are particulars or are proposition-like objects. Jaegwon Kim, a principal proponent of the latter view, also challenges the view that mathematical objects are “causally inert”: Mathematical properties, including numbers, are no worse off than such sundry physical properties as color, mass, and volume, in respect of causal efficacy (“The Role of Perception,” p. 347).

It is noteworthy that Kim talks of mathematical properties. The point could be put by saying that reference to mathematical objects is as likely to occur in a causal explanation as reference to other kinds of objects. This seems to me quite correct, but, as Mark Steiner had previously observed (Mathematical Knowledge, ch. 4), it can be put in the other framework, in which such reference will occur in the relevant descriptions of the events.
It is not essential for our purposes that there should be a principled and exhaustive classification of all objects into abstract and concrete. Physical bodies and biological organisms, such as we encounter in everyday life, are concrete. If we assume that sense-perception necessarily involves a causal relation between the object perceived and the organism (the event or state of its perceiving), and that perception locates its objects at least in some rough way, then it follows that the objects of sense-perception are concrete. Thus it is generally assumed in discussions of abstract objects that abstract objects cannot be perceived by the senses.

In this they are not alone. It is not merely for this reason that abstract objects are thought to pose a general philosophical problem. Some presumably concrete objects, such as elementary particles, are far more recondite and no more perceivable than most “ordinary” abstract objects such as triangles and numbers. Nonetheless, some form of empiricism is a primary motive for finding abstract objects puzzling. We think there are elementary particles such as electrons, roughly because the assumption that they exist belongs to a theory that presents a picture of the objective world that accounts for the facts we can verify by perception. But why should a view of the world give place to objects that are not in it (not spatio-temporal) and do not interact with it? Is the existence of mathematical objects not a hypothesis for which we should have no need?

If the outcome of an investigation of such questions should be that mathematics is engaged in mythology, in speaking of remarkable configurations of objects that just do not exist, it would leave the existence of mathematics and its importance to science and practical life a much greater mystery than its objects are at the outset. This would suggest that what such questions should prompt us to do is to inquire more deeply into what a mathematical object is, what the “mode of being” of mathematical objects is. Or, in less ontological terms, we should inquire into the sense in which mathematics speaks of objects. This will be one of the main concerns of this book. It is, however, a serious question whether reference to specifically mathematical objects can be eliminated. This question will be addressed in Chapters 2 and 3 below. Assuming that a wholesale rejection of mathematics is not being considered, such an elimination can only proceed by not taking the language of mathematics entirely at face value.

standing in causal relations. This removes any temptation to attribute causal efficacy to the mathematical objects. By his emphasis on properties, Kim somewhat fudges the latter issue.

This question will prove to have relevance later; see §30.
There is something absurd about inquiring, with complete generality, what an *object* is. The question may seem impossibly amorphous, but it is the extreme generality of the question, rather than vagueness in the ordinary sense, or something like it, that is the difficulty in putting it. If we ask what a gorilla is, an informative answer should distinguish gorillas from other animals. But from what can we “distinguish” objects? It is controversial whether it is meaningful to talk of “entities” that are not objects, and, even if it is, a distinction between objects and other entities is only part of what we are after in asking what an object is.

To the extent that philosophers discuss in general terms what an object is, the context is likely to be an inquiry into how thought or language can relate to objects, or, more amorphously, to “reality” or “the world.” I will fix the way I wish to use the term “object” and simultaneously say what I think useful in such abstract discussions by saying that the usable general characterization of the notion of object comes from *logic*. We speak of particular objects by referring to them by singular terms: names, demonstratives, and descriptions. Since the unit of linguistic expression is the sentence, reference to objects will typically and standardly occur in the context of sentences, and, therefore, in the use of singular terms, by the application to singular terms of *predicates*. In its most general meaning, a “predicate” is a sentence with one or more empty places, called argument places, to be filled by singular terms.

Reference to objects may begin with singular terms, but it does not end with them, since the use of predicates is bound up with the use of those expressions which the formal logician regiments by quantifiers and bound variables. Thus, in English and many other natural languages, what I have called the “argument place” of a predicate can be filled by expressions like ‘all men’, ‘every man’, ‘all ravens’, ‘everything’, ‘something’, ‘sometime’, and others which serve to make the sentence general, or by pronouns that refer back either to one of these expressions or to a singular term. Thus, we have such sentences or discourses as:

All men are mortal. All ravens are black.
Everything is identical to itself.
Do something!
Come around to see me sometime.
If anyone know just cause why this man and this woman should not be joined in holy matrimony, let him speak now or forever hold his peace.

I saw Mr. Smith on January 10. He was in a bad temper that day.

Philosophers in the tradition of the modern theory of reference have disagreed about whether singular terms or quantifiers and bound variables are the more fundamental vehicles of reference to objects. It is quite evident that in practice we have to attend to both kinds of expressions. Singular terms may be dispensable in a final regimentation of the language of a rigorous science or mathematics, but, in order to understand reference and the notion of object, we have to look at language as it is before such regimentation, in which singular terms have an undoubted place. It is even more obvious that the use of singular terms does not exhaust reference to objects in the sense that the objects whose properties and relations matter to the truth of what we say are not all referred to by singular terms, either in the statements themselves, or in any others. Our ontological commitments, that is, what is required to exist for our statements to be true, are not all expressed by the use of singular terms but are also embodied in general statements; indeed, a definite description (at least in its “attributive” use) is a singular term such that the condition for being its reference is the truth of a statement involving a quantifier.

The phrase “concept of an object in general” comes, of course, from Kant’s *Critique of Pure Reason*. It is used in connection with the categories, in particular with the thesis that the categories are the conditions under which what is given in experience can be thought of as an object. For example,

> The question now is whether *a priori* concepts do not also precede, as conditions under which alone something can be, if not intuited, nevertheless thought as object in general. (A 93/B 125)

Kant calls the categories “concepts of objects in general,” but he also occasionally talks as if there were one concept of an object in general, that

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2 I assume that bound variables (and the pronouns of natural language that play the same role) do not count as singular terms. I don’t think it matters where we draw the line, though with enough stretching of the category it might be made true, after all, that all objects of reference for our discourse are denoted by singular terms, since the semantics of quantifiers refers to assignments of objects to free variables or to interpretations in which the denotation of names is varied.

3 In the well-known sense of Donnellan, “Reference and Definite Descriptions.”
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is, a perfectly general concept of object. But then it is the categories that give it its content, so that the difference with the plural use is merely verbal. Either way, Kant is thinking of objects of experience, and the proper application of the categories is to spatiotemporal objects of the sort that are given in perception or at least postulated in scientific theories.

In fact, when Kant talks of objects at this level of abstractness and generality, he is pulled in two directions. On the one hand, the categories are derived from formal logic and from the conceptual framework for talking of any kind of object, whether or not it is empirically known. The understanding, with its forms of judgment and categories, is more general than sensibility, with its forms of intuition, and thus there are “pure” categories whose content relates to objects “in general and in themselves,” and are not limited by conditions of sensibility or experience.

On the other hand, it is of course central to Kant’s theory of knowledge that the categories only have proper application giving rise to knowledge in application to the manifold given by sensible intuition, and thus they serve only for knowledge of the empirically given world. It is only in this context that he gives an informative account of consciousness of objects. And of course it has been difficult for many generations of readers to see how the application, even “problematic,” of the categories to things in themselves is possible.

Even the pure categories, as a general conceptual apparatus for speaking of objects, envisage concrete rather than abstract objects. This is particularly true of the categories of relation: substance, causality, and community. We gave as one of the marks of an abstract object its not entering into causal relations. Although some marks of a substance may be possessed by abstract objects, the primary examples of substances in the philosophical systems of the past are paradigmatically concrete realities: organisms for Aristotle, God for Christian Aristotelians and the rationalists from Descartes on, and matter for Descartes and Kant. When Kant comes to the categories of modality, what in the original table (A 80/B 106) is called Existence (Dasein) and is later called actuality (Wirksamkeit) is explained in a way which clearly is meant to apply to concrete spatiotemporal objects:

That which is connected with the material conditions of experience (of sensation) is actual. (A 218/B 266)

4 A 51 / B 75, A 251. The latter passage seems to identify the concept of an object in general with that of the “transcendental object = x.”
The postulate for cognizing the actuality of things requires perception, thus sensation of which one is conscious, to be sure not immediate perception of the object itself the existence of which is to be cognized, but still its connection with some actual perception in accordance with the analogies of experience. (A 225/B 272)

By actuality, Kant means actual existence, and it therefore has the logical properties which, for existence per se, he appeals to in his famous criticism of proofs of the existence of God. It is not a property of objects; we could reconstruct it as what is expressed by the existential quantifier in empirical judgments. But the characterization quoted gives just what we have offered above as the marks of a concrete object.

In our modern logic, where we readily give the existential quantifier a more general sense, it is natural to render Kant's actuality as a restricted quantifier, so that ‘Fs are actual’ in Kant's sense would be rendered ‘(∃x)(x is actual ∧Fx)’. The German ‘wirklich’ is less awkward as a predicate than the English ‘actual’.

Just such a transmutation is carried out by Frege, who denies that numbers are actual (wirklich) in a sense which (probably consciously) echoes Kant:

I heartily share his [Cantor's] contempt for the view that in principle only finite numbers ought to be admitted as actual. Perceptible by the senses they are not, nor are they spatial – any more than fractions are, or negative numbers, or irrational and complex numbers; if we restrict the actual to what acts on our senses or at least produces effects which may cause sense-perceptions as near or remote consequences, then naturally no number of any of these kinds is actual.5

Frege clearly understands Wirklichkeit as a property which some objects (such as bodies) have and others (such as numbers) do not.6 In using the terms ‘actual’ and ‘wirklich’, I will follow Frege’s usage.

5 Grundlagen, p. 97.
6 As in the following striking passage:

Some of what is objective is actual, other not. ‘Actual’ is only one of many predicates, and concerns logic no more particularly than, say, the predicate ‘algebraic’ applied to a curve (Grundgesetze, I, xviii–xix, my trans.).

Frege seems pretty clearly to have the same meaning of ‘wirklich’ in mind when, near the end of “Der Gedanke,” he questions whether thoughts are wirklich. The strict atemporality of thoughts is “annulled” by the fact that they can be now grasped by one person, at another time not. However, one might still recognize them as timeless on the ground that these changes affect only the “inessential” properties of thoughts. (We might now call them “mere Cambridge changes.”) Moreover, he is prepared to say that a
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To return to Kant, I think it would be fair to interpret him as having a “concept of an object in general” which in its full sense connotes concreteness. This is even so in the less-than-full sense given by the pure categories. A problem in interpreting Kant is posed by the fact that, although he does make reference to mathematical objects, and is certainly not a nominalist, he does not have an explicit theory of them. There are indications that he would treat mathematical existence under the category of possibility; mathematical examples occur in the elucidation of that category in the Postulates. There is some affinity between Kant’s view and those now called modalist, which will occupy us in Chapter 3. This affinity is limited by the fact that Kant conceives possibility as “real” possibility, not certifiable by mathematical argument (construction in pure intuition) alone. It would take us too far afield to go into these problems, which I have discussed elsewhere.

The expectation that objects as such should have the properties that Kant and Frege associate with *Wirksamkeit*, or other more rarefied versions that could be possessed by suprasensible concrete reality, is deeply rooted and motivates much resistance to mathematical objects and abstract objects generally. Taking the most general notion of object to have its home in formal logic (in Kantian terms, the forms of judgment rather than the categories) is intended to remove that expectation and serves to defuse the resistance it motivates. This perspective will underlie much of our subsequent argument.

Kant’s refusal to apply the category of actuality to mathematical objects suggests another distinctive feature of such objects, that the distinction between potentiality and actuality, particularly between possible and thought “acts” (*wirkt*) through being grasped and taken to be true. But because it is only in this way that thoughts enter into the causal order, the “actuality” of thoughts is very different from that of things:

Thoughts are not wholly unactual, but their actuality is of a quite different kind from that of things. And their action is brought about by a performance of the thinker, without which they would be without effect (*wirkungslos*), at least as far as we can see. And yet the thinker does not create them but must take them as they are (p. 77; trans. from *Logical Investigations*, pp. 29–30, with modifications).

I am indebted to Mark Notturno for calling this tantalizing passage to my attention. In it Frege shows himself more properly “Platonist” than in his writings on the philosophy of mathematics.

8 “Arithmetic and the Categories,” section I, and, in a briefer and more preliminary way, in the Postscript to Essay 5 of *Mathematics in Philosophy*, pp. 147–149.
actual existence, does not apply to them. This is an important point, which will be considered briefly in §4 below and more fully in Chapters 2 and 3.

§3. Intuitability

Although Kant would resist attributing to mathematical objects the properties summed up in the concept of actuality, he does apply in the mathematical context another characteristic of full-blooded empirical objects which will be important in what follows. Kant of course holds that a concept is empty unless it corresponds to intuition; intuition is necessary to establish the objective reality of a concept, that is, the possibility of instances. The forms of intuition, space and time, are therefore conditions to which all objects of experience must conform. The mathematical objects about which Kant is most explicit are geometric figures, which he calls forms of (empirical) objects. In proofs, they are constructed intuitively; in that sense they can be intuited. Intuitive representation also arises in mathematics for other mathematical objects, though for numbers in particular it appears that the relation to intuition is more indirect than for geometric figures. But arithmetic is according to Kant only applicable to sensible objects, that is, to objects given according to our forms of intuition.

We will speak, generally and somewhat vaguely, of intuitability as a general condition on objects. The use of the term “intuition” rather than, say, “perception” is meant to preserve the generality of Kant’s notion, which, in particular, is not meant to exclude the abstract. Kant means by “intuition” an immediate representation of an individual object. What is meant by “immediate” has been a matter of controversy. When talking of intuition of objects in this work, I shall mean a mode of consciousness of individual objects that is importantly analogous to perception. But it is important to distinguish between intuition of objects and the different

9 In 1790 Kant’s disciple Johann Schultz wrote that “in mathematics possibility and actuality are one, and the geometer says there are (es gibt) conic sections, as soon as he has shown their possibility a priori, without inquiring as to the actual drawing or making them from material” (Review of vol. II of Eberhard’s Philosophisches Magazin, in Kant’s Gesammelte Schriften, XX 386 n.) (Cf. “Arithmetic and the Categories,” p. 110 and n. 4.)

10 Letter to Johann Schultz, November 25, 1788 (Gesammelte Schriften, X 555). This edition is hereafter referred to as Ak.

11 See Essay 5 of Mathematics in Philosophy, especially the Postscript, the writings of Hintikka and Howell referred to there, and “The Transcendental Aesthetic,” section I.
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but related notion of intuition as a propositional attitude. These notions and their relations will be explored further in Chapter 5. Clearly it is the former notion that is involved when we speak of the intuitability of objects.

Even if one takes the notion of intuition to be understood, the notion of intuitability poses two serious problems. As we have explained it, intuitability does not require that the object itself can be an object of intuition. But if it is, we might call the object strongly intuitable. We consider intuitable an object that can be “represented” in intuition, and we have not said what this means. Different relations will count as such representation; thus the notion of intuitability is to a certain degree schematic. In the relevant senses, however, representation of abstract objects by concrete objects or by objects relatively closer to the concrete is a pervasive phenomenon and of great importance for understanding abstract objects. An example is that a number could be represented by a set or sequence of that number of objects; such a representation occurs in Kant’s argument to the effect that $7 + 5 = 12$ is synthetic. If, as in Kant’s example, the elements of the set or the terms of the sequence are concrete, then the representation is what in §7 will be called “quasi-concrete.” We will consider in Chapter 6 the question whether, if they are intuitable, the set or sequence itself is intuitable. If so, then the representation certainly qualifies as a representation in intuition; even if not, it is still a representation in intuition of a more indirect kind, since the concrete is reached in two steps. If such representation confers intuitability, then mathematical objects can count as intuitable even if no mathematical objects are strongly intuitable. This may seem to make the notion too weak to be useful; whether this is so will be considered in Chapters 5 and 6.

The second problem concerns the modal element in the notion of intuitability. We might call an object “perceivable” if it can be perceived, but this “can” can be understood in different ways; for example, to be precise we would have to determine how much to abstract from the situation and capacities of actual perceivers. Questions of this kind arise with particular force when we ask about the intuitability of mathematical objects. Is it true that all natural numbers can be represented in intuition in the sense that for any $n$, a set of $n$ objects can be an object of intuition? That seems to require that in some sense we “can” take in an arbitrarily large finite collection, but, in practice, one will very quickly run up against limitations of time and memory. Classical conceptions of mathematical intuition found in Kant, Brouwer, and Hilbert, by contrast, appear to be committed to the view that in some way arbitrarily large, finite structures
are representable in intuition. The viability of the idea of the “in principle” possibility of intuition's extending this far, and the question whether a conception of mathematical intuition along such lines requires it, will occupy us later.

One can ask how the notions of intuitability and actuality are related. The thesis that all intuitable objects are actual would rule out intuition of mathematical objects, but it is certainly maintained by many philosophers. It is interesting only if the notion of intuitability is either strong intuitability or based on a notion of representation in intuition that is not too liberal. The converse thesis that all actual objects are intuitable amounts to the thesis that those objects we postulate in a causal account of our perceptions are intuitable. Here, clearly, the question about modality is relevant, and also questions about the “theory-laden” character of observation.

§4. Logic and the notion of object

I have mentioned actuality and intuitability not just because of their centrality in Kant's conception of an object of experience. They represent requirements for being an object that have been applied by philosophers, in some cases not very consciously, to issues about abstract objects and inclined them to deny that there could be such objects, or, at least, to find them puzzling. The view that the most general notion of object has its home in formal logic is intended to reject such conditions as conditions for being an object.

This outlook seems to me a characteristically modern one and may not appear in full-blown form before Frege. Its most important advocates in more recent times are Carnap and Quine. Speaking of objects just is using the linguistic devices of singular terms, predication, identity, and quantification to make serious statements. We might in many cases be able to claim that the reference to objects in a discourse is only apparent. But this requires either a paraphrase or a special semantical explanation that removes it; one cannot, for example, just say that the objects in question are fictions. One might explain “fictions” as a category of objects, but in that case one has not fended off the interpretation that one is speaking of objects.12

12 With reference to mathematics, one might also mean by such a claim that although the language of mathematics is to be interpreted to speak of objects, the mathematician is not committed to their existence because the statements in mathematics are not really