PART 1 GRAVITATION AND RELATIVITY

1 Essentials of general relativity

1.1 The concepts of general relativity

SPECIAL RELATIVITY To understand the issues involved in general relativity, it is helpful to begin with a brief summary of the way space and time are treated in special relativity. The latter theory is an elaboration of the intuitive point of view that the properties of empty space should be the same throughout the universe. This is just a generalization of everyday experience: the world in our vicinity looks much the same whether we are stationary or in motion (leaving aside the inertial forces experienced by accelerated observers, to which we will return shortly).

The immediate consequence of this assumption is that any process that depends only on the properties of empty space must appear the same to all observers: the velocity of light or gravitational radiation should be a constant. The development of special relativity can of course proceed from the experimental constancy of c, as revealed by the Michelson-Morley experiment, but it is worth noting that Einstein considered the result of this experiment to be inevitable on intuitive grounds (see Pais 1982 for a detailed account of the conceptual development of relativity). Despite the mathematical complexity that can result, general relativity is at heart a highly intuitive theory; the way in which our everyday experience can be generalized to deduce the large-scale structure of the universe is one of the most magical parts of physics. The most important concepts of the theory can be dealt with without requiring much mathematical sophistication, and we begin with these physical fundamentals.

4-VECTORS From the constancy of c, it is simple to show that the only possible linear transformation relating the coordinates measured by different observers is the Lorentz transformation:

$$\boxed{\begin{aligned} dx' &= \gamma \left(dx - \frac{v}{c} c\, dt \right) \\ c\, dt' &= \gamma \left(c\, dt - \frac{v}{c} dx \right). \end{aligned}} \tag{1.1}$$

Note that this is written in a form that makes it explicit that x and ct are treated in the same way. To reflect this interchangeability of space and time, and the absence of any preferred frame, we say that special relativity requires all true physical relations to be written in terms of **4-vectors**. An equation valid for one observer will then apply to all others because the quantities on either side of the equation will transform in the same way. We ensure that this is so by constructing physical 4-vectors out of the fundamental interval

$$dx^\mu = (c\,dt, dx, dy, dz) \quad \mu = 0, 1, 2, 3, \tag{1.2}$$

by manipulations with relativistic invariants such as rest mass m and proper time $d\tau$, where

$$(c\,d\tau)^2 = (c\,dt)^2 - (dx^2 + dy^2 + dz^2). \tag{1.3}$$

Thus, defining the 4-momentum $P^\mu = m\,dx^\mu/d\tau$ allows an immediate relativistic generalization of conservation of mass and momentum, since the equation $\Delta P^\mu = 0$ reduces to these laws for an observer who sees a set of slowly moving particles. This is a very powerful principle, as it allows us to reject 'obviously wrong' physical laws at sight. For example, Newton's second law $\mathbf{F} = m\,d\mathbf{u}/dt$ is not a relation between the spatial components of two 4-vectors. The obvious way to define 4-force is $F^\mu = dP^\mu/d\tau$, but where does the 3-force $\mathbf{F}$ sit in F^μ? Force will still be defined as rate of change of momentum, $\mathbf{F} = d\mathbf{P}/dt$; the required components of F^μ are $\gamma(\dot{E}, \mathbf{F})$, and the correct relativistic force–acceleration relation is

$$\mathbf{F} = m\frac{d}{dt}(\gamma\mathbf{u}). \tag{1.4}$$

Note again that the symbol m denotes the **rest mass** of the particle, which is one of the invariant scalar quantities of special relativity. The whole ethos of special relativity is that, in the frame in which a particle is at rest, its intrinsic properties such as mass are always the same, independently of how fast it is moving. The general way in which quantities are calculated in relativity is to evaluate them in the rest frame where things are simple, and then to transform out into the lab frame.

GENERAL RELATIVITY Nothing that has been said so far seems to depend on whether or not observers move at constant velocity. We have in fact already dealt with the main principle of general relativity, which states that the only valid physical laws are those that equate two quantities that transform in the same way under any arbitrary change of coordinates.

Before getting too pleased with ourselves, we should ask how we are going to construct general analogues of 4-vectors. Consider how the components of dx^μ transform under the adoption of a new set of coordinates x'^μ, which are functions of x^ν:

$$\boxed{dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}dx^\nu.} \tag{1.5}$$

This apparently trivial equation (which assumes, as usual, the summation convention on repeated indices) may be divided by $d\tau$ on either side to obtain a similar transformation law for 4-velocity, U^μ; so U^μ is a general 4-vector. Things unfortunately go wrong at the next level, when we try to differentiate this new equation to form the 4-acceleration $A^\mu = dU^\mu/d\tau$:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}A^\nu + \frac{\partial^2 x'^\mu}{\partial\tau\,\partial x^\nu}U^\nu. \tag{1.6}$$

The second term on the right-hand side (rhs) is zero only when the transformation coefficients are constants. This is so for the Lorentz transformation, but not in general. The conclusion is therefore that $F^\mu = dP^\mu/d\tau$ cannot be a general law of physics, since $dP^\mu/d\tau$ is not a general 4-vector.

INERTIAL FRAMES AND MACH'S PRINCIPLE We have just deduced in a rather cumbersome fashion the familiar fact that $\mathbf{F} = m\mathbf{a}$ only applies in inertial frames of reference. What exactly are these? There is a well-known circularity in Newtonian mechanics, in that inertial frames are effectively defined as being those sets of observers for whom $\mathbf{F} = m\mathbf{a}$ applies. The circularity is only broken by supplying some independent information about $\mathbf{F}$ – for example, the Lorentz force $\mathbf{F} = e(\mathbf{E} + \mathbf{v}\wedge\mathbf{B})$ in the case of a charged particle. This leaves us in a rather unsatisfactory situation: $\mathbf{F} = m\mathbf{a}$ is really only a statement about cause and effect, so the existence of non-inertial frames comes down to saying that there can be a motion with no apparent cause. Now, it is well known that $\mathbf{F} = m\mathbf{a}$ can be made to apply in all frames if certain 'fictitious' forces are allowed to operate. In respectively uniformly accelerating and rotating frames, we would write

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} + m\mathbf{g} \\ \mathbf{F} &= m\mathbf{a} + m\mathbf{\Omega}\wedge(\mathbf{\Omega}\wedge\mathbf{r}) - 2m(\mathbf{v}\wedge\mathbf{\Omega}) + m\dot{\mathbf{\Omega}}\wedge\mathbf{r}. \end{aligned} \tag{1.7}$$

The fact that these 'forces' have simple expressions is tantalizing: it suggests that they should have a direct explanation, rather than taking the Newtonian view that they arise from an incorrect choice of reference frame. The relativist's attitude will be that if our physical laws are correct, they should account for what observers see from any arbitrary point of view – however perverse.

The mystery of inertial frames is deepened by a fact of which Newton was well aware, but did not explain: an inertial frame is one in which the bulk of matter in the universe is at rest. This observation was taken up in 1872 by Ernst Mach. He argued that since the acceleration of particles can only be measured relative to other matter in the universe, the existence of inertia for a particle must depend on the existence of other matter. This idea has become known as **Mach's principle**, and was a strong influence on Einstein in formulating general relativity. In fact, Mach's ideas ended up very much in conflict with Einstein's eventual theory – most crucially, the rest mass of a particle is a relativistic invariant, independent of the gravitational environment in which a particle finds itself. However, controversy still arises in debating whether general relativity is truly a 'Machian' theory – i.e. one in which the rest frame of the large-scale matter distribution is inevitably an inertial frame (e.g. Raine & Heller 1981).

A hint at the answer to this question comes by returning to the expressions for the inertial forces. The most satisfactory outcome would be to dispose of the notion of inertial frames altogether, and to find a direct physical mechanism for generating 'fictitious' forces. Following this route in fact leads us to conclude that Newtonian gravitation cannot be correct, and that the inertial forces can be effectively attributed to gravitational radiation. Since we cannot at this stage give a correct relativistic argument, consider the analogy with electromagnetism. At large distances, an accelerating charge produces an electric field given by

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0 r c^2}(\hat{\mathbf{r}}\wedge[\mathbf{a}])\wedge\hat{\mathbf{r}}, \tag{1.8}$$

i.e. with components parallel to the retarded acceleration $[\mathbf{a}]$ and perpendicular to the acceleration axis. A charge distribution symmetric about a given point will then generate a net force on a particle at that point in the direction of **a**. It is highly plausible that something similar goes on in the generation of inertial forces via gravity, and we can guess the magnitude by letting $e/(4\pi\epsilon_0) \to Gm$. This argument was proposed by Dennis Sciama, and is known as **inertial induction**. Integrating such a force over all mass in a spherically symmetric universe, we get a total of

$$\frac{F_{\rm tot}}{m} = 2\pi \frac{Ga}{c^2} \int_0^{c/H_0} \int_0^{\pi} \rho\, r\, \sin^3\theta\; d\theta\; dr\; = a\, \frac{\pi^2 G\rho}{2H_0^2}. \tag{1.9}$$

This calculation is rough in many respects. The main deficiency is the failure to include the expansion of the universe: objects at a vector distance **r** appear to recede from us at a velocity $\mathbf{v} = H_0\mathbf{r}$, where H_0 is known as Hubble's constant (and is not constant at all, as will become apparent later). This law is only strictly valid at small distances, of course, but it does tell us that objects with $r \simeq c/H_0$ recede at a speed approaching that of light. This is why it seems reasonable to use this as an upper cutoff in the radial part of the above integral. Having done this, we obtain a total acceleration induced by gravitational radiation that is roughly equal to the acceleration we first thought of (the dimensionless factor on the rhs of the above equation is known experimentally to be unity to within a factor 10 or so). Thus, it does seem qualitatively valid to think of inertial forces as arising from gravitational radiation. Apart from being a startlingly different view of what is going on in non-inertial frames, this argument also sheds light on Mach's principle: for a symmetric universe, inertial forces clearly vanish in the average rest frame of the matter distribution. Frames in constant relative motion are allowed because (in this analogy) a uniformly moving charge does not radiate.

It is not worth trying to make this calculation more precise, as the approach is not really even close to being a correct relativistic treatment. Nevertheless, it does illustrate very well the prime characteristic of relativistic thought: we must be able to explain what we see from any point of view.

THE EQUIVALENCE PRINCIPLE In the previous subsection, we were trying to understand the non-inertial effects that are seen in accelerating reference frames as being gravitational in origin. In fact, it is more conventional to state this equivalence the other way around, saying that gravitational effects are identical in nature to those arising through acceleration. The seed for this idea goes back to the observation by Galileo that bodies fall at a rate independent of mass. In Newtonian terms, the acceleration of a body in a gravitational field **g** is

$$m_{\rm I}\, \mathbf{a} = m_{\rm G}\, \mathbf{g}, \tag{1.10}$$

and no experiment has ever been able to detect a difference between the inertial and gravitational masses $m_{\rm I}$ and $m_{\rm G}$ (the equality holds to better than 1 part in 10^{11}: Will 1993). This equality is trivially obvious in the case of inertial forces, and the apparent gravitational acceleration **g** becomes simply the acceleration of the frame **a**. These considerations led Einstein to suggest that inertial and gravitational forces were indeed one and the same. Formally, this leads us to the equivalence principle, which comes in two forms.

The **weak equivalence principle** is a statement only about space and time. It says that in any gravitational field, however strong, a freely falling observer will experience no gravitational effects – with the important exception of tidal forces in non-uniform fields. The spacetime will be that of special relativity (known as **Minkowski spacetime**).

The **strong equivalence principle** takes this a stage further and asserts that not only is the spacetime as in special relativity, but all the laws of physics take the same form in the freely falling frame as they would in the absence of gravity. This form of the equivalence principle is crucial in that it will allow us to deduce the generally valid laws governing physics once the special-relativistic forms are known. Note however that it is less easy to design experiments that can *test* the strong equivalence principle (see chapter 8 of Will 1993).

It may seem that we have actually returned to something like the Newtonian viewpoint: gravitation is merely an artifact of looking at things from the 'wrong' point of view. This is not really so; rather, the important aspects of gravitation are not so much to do with first-order effects as second-order tidal forces: these cannot be transformed away and are the true signature of gravitating mass. However, it is certainly true in one sense to say that gravity is *not* a real force: the gravitational acceleration is not derived from a 4-force F^μ and transforms differently.

GRAVITATIONAL TIME DILATION Many of the important features of general relativity can be obtained via rather simple arguments that use the equivalence principle. The most famous of these is the thought experiment that leads to gravitational time dilation, illustrated in figure 1.1. Consider an accelerating frame, which is conventionally a rocket of height h, with a clock mounted on the roof that regularly disgorges photons towards the floor. If the rocket accelerates upwards at g, the floor acquires a speed $v = gh/c$ in the time taken for a photon to travel from roof to floor. There will thus be a blueshift in the frequency of received photons, given by $\Delta\nu/\nu = gh/c^2$, and it is easy to see that the rate of reception of photons will increase by the same factor.

Now, since the rocket can be kept accelerating for as long as we like, and since photons cannot be stockpiled anywhere, the conclusion of an observer on the floor of the rocket is that in a real sense the clock on the roof is running fast. When the rocket stops accelerating, the clock on the roof will have gained a time Δt by comparison with an identical clock kept on the floor. Finally, the equivalence principle can be brought in to conclude that gravity must cause the same effect. Noting that $\Delta\phi = gh$ is the difference in potential between roof and floor, it is simple to generalize this to

$$\boxed{\frac{\Delta t}{t} = \frac{\Delta\phi}{c^2}.} \tag{1.11}$$

The same thought experiment can also be used to show that light must be deflected in a gravitational field: consider a ray that crosses the rocket cabin horizontally when stationary. This track will appear curved when the rocket accelerates.

The experimental demonstration of the gravitational redshift by Pound & Rebka (1960) was one of the main pieces of evidence for the essential correctness of the above reasoning, and provides a test (although not the most powerful one) of the equivalence principle.

THE TWIN PARADOX One of the neatest illustrations of gravitational time dilation is in resolving the twin paradox. This involves twins A and B, each equipped with a clock.

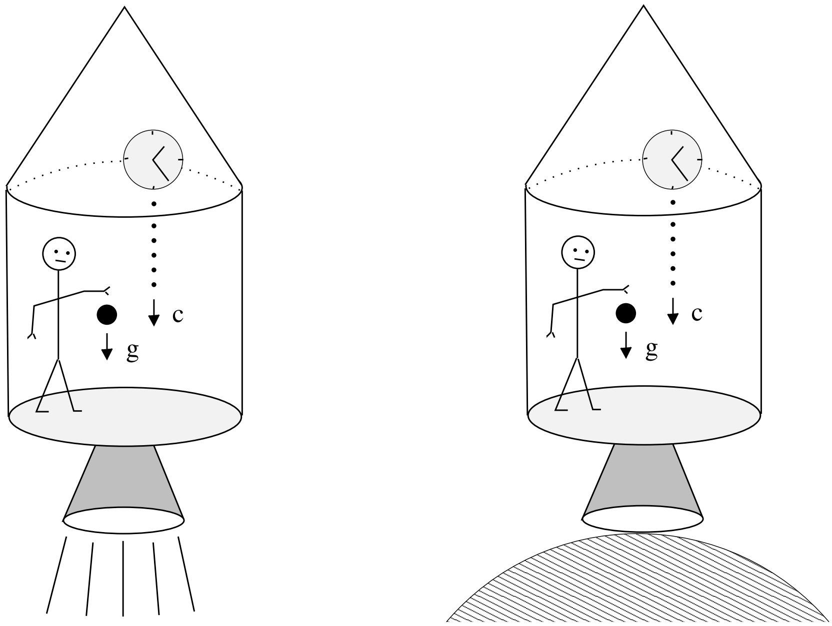

Figure 1.1. Imagine you are in a box in free space far from any source of gravitation. If the box is made to accelerate 'upwards' and has a clock that emits a photon every second mounted on its roof, it is easy to see that you will receive photons more rapidly once the box accelerates (imagine yourself running into the line of oncoming photons). Now, according to the equivalence principle, the situation is exactly equivalent to the second picture in which the box sits at rest on the surface of the Earth. Since there is nowhere for the excess photons to accumulate, the conclusion has to be that clocks above us in a gravitational field run fast.

A remains on Earth, while B travels a distance d on a rocket at velocity v, fires the engines briefly to reverse the rocket's velocity, and returns. The standard analysis of this situation in special relativity concludes, correctly, that A's clock will indicate a longer time for the journey than B's:

$$t_{\rm A} = \gamma\, t_{\rm B}. \tag{1.12}$$

The so-called paradox lies in the broken symmetry between the twins. There are various resolutions of this puzzle, but these generally refuse to meet the problem head-on by analysing things from B's point of view. However, at least for small v, it is easy to do this using the equivalence principle. There are three stages to consider:

(1) Outward trip. According to B, in special relativity A's clock runs slow: $t_{\rm A} = \gamma^{-1} t_{\rm B} \simeq [1 - v^2/(2c^2)](d/v)$.

(2) Return trip. Similarly, A's clock runs slow, resulting in a total lag with respect to B's of $(v^2/c^2)(d/v) = vd/c^2$.

(3) In between comes the crucial phase of turning. During this time, B's frame is non-inertial; there is an apparent gravitational field causing A to halt and start to return to B (at least, what else is B to conclude? There is obviously a force acting on the Earth, but the Earth is clearly not equipped with rockets). If an acceleration g operates for a time $t_{\rm turn}$, then A's clock will run fast by a fractional amount gd/c^2, leading to a total time step of $gdt_{\rm turn}/c^2 = 2vd/c^2$ (since $gt_{\rm turn} = 2v$).

Thus, in total, B returns to find A's clock in advance of B's by an amount

$$t_{\rm A} - t_{\rm B} = -\frac{vd}{c^2} + \frac{2vd}{c^2} \simeq (\gamma - 1)t_{\rm B}, \tag{1.13}$$

exactly (for small v) in accordance with A's entirely special relativity calculation.

1.2 The equation of motion

It was mentioned above that the equivalence principle allows us to bootstrap our way from physics in Minkowski spacetime to general laws. We can in fact obtain the full equations of general relativity in this way, in an approach pioneered by Weinberg (1972). In what follows, note the following conventions: Greek indices run from 0 to 3 (spacetime), Roman from 1 to 3 (spatial). The summation convention on repeated indices of either type is assumed.

Consider freely falling observers, who erect a special-relativity coordinate frame ξ^μ in their neighbourhood. The equation of motion for nearby particles is simple:

$$\frac{d^2\xi^\mu}{d\tau^2} = 0; \qquad \xi^\mu = (ct, x, y, z), \tag{1.14}$$

i.e. they have zero acceleration, and we have Minkowski spacetime

$$c^2 d\tau^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta, \tag{1.15}$$

where $\eta_{\alpha\beta}$ is just a diagonal matrix $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$. Now suppose the observers make a transformation to some other set of coordinates x^μ. What results is the perfectly general relation

$$d\xi^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} dx^\nu, \tag{1.16}$$

which on substitution leads to the two principal equations of dynamics in general relativity:

$$\boxed{\begin{aligned} \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} &= 0 \\ c^2 d\tau^2 &= g_{\alpha\beta}\, dx^\alpha\, dx^\beta. \end{aligned}} \tag{1.17}$$

At this stage, the new quantities appearing in these equations are defined only in terms of our transformation coefficients:

$$\begin{aligned} \Gamma^\mu_{\alpha\beta} &= \frac{\partial x^\mu}{\partial \xi^\nu}\frac{\partial^2 \xi^\nu}{\partial x^\alpha \partial x^\beta} \\ g_{\mu\nu} &= \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}. \end{aligned} \tag{1.18}$$

COORDINATE TRANSFORMATIONS What is the physical meaning of this analysis? We have taken the special relativity equations for motion and the structure of spacetime and looked at the effects of a general coordinate transformation. One example of such a transformation is a Lorentz boost to some other inertial frame. However, this is not very interesting since we know in advance that the equations retain their form in this case (it is easy to show that $\Gamma^{\mu}_{\alpha\beta} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$). A more general transformation could be one to the frame of an accelerating observer, but the transformation might have *no* direct physical interpretation at all. It is important to realize that general relativity makes no distinction between coordinate transformations associated with motion of the observer and a simple change of variable. For example, we might decide that henceforth we will write down coordinates in the order (x, y, z, ct) rather than (ct, x, y, z) (as is indeed the case in some formalisms). General relativity can cope with these changes automatically. Indeed, this flexibility of the theory is something of a problem: it can sometimes be hard to see when some feature of a problem is 'real', or just an artifact of the coordinates adopted. People attempt to distinguish this second type of coordinate change by distinguishing between 'active' and 'passive' Lorentz transformations; a more common term for the latter class is **gauge transformation**. The term gauge will occur often throughout this book: it always refers to some freedom within a theory that has no observable consequence (e.g. the arbitrary value of $\boldsymbol{\nabla} \cdot \mathbf{A}$, where $\mathbf{A}$ is the vector potential in electrodynamics).

METRIC AND CONNECTION The matrix $g_{\mu\nu}$ is known as the **metric tensor**. It expresses (in the sense of special relativity) a notion of distance between spacetime points. Although this is a feature of many spaces commonly used in physics, it is easy to think of cases where such a measure does not exist (for example, in a plot of particle masses against charges, there is no physical meaning to the distance between points). The fact that spacetime *is* endowed with a metric is in fact something that has been *deduced*, as a consequence of special relativity and the equivalence principle. Given a metric, Minkowski spacetime appears as an inevitable special case: if the matrix $g_{\mu\nu}$ is symmetric, we know that there must exist a coordinate transformation that makes the matrix diagonal:

$$\tilde{\boldsymbol{\Lambda}}\mathbf{g}\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_0, \ldots, \lambda_3), \qquad (1.19)$$

where $\boldsymbol{\Lambda}$ is the matrix of transformation coefficients, and λ_i are the eigenvalues of this matrix.

The object $g_{\mu\nu}$ is called a **tensor**, since it occurs in an equation $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ that must be valid in all frames. In order for this to be so, the components of the matrix $\mathbf{g}$ must obey certain transformation relations under a change of coordinates. This is one way of defining a tensor, an issue that is discussed in detail below.

So much for the metric tensor, what is the meaning of the coefficients $\Gamma^{\mu}_{\alpha\beta}$? These are known as components of the **affine connection** or as **Christoffel symbols** (and are sometimes written in the alternative notation $\left\{ {\mu \atop \alpha\beta} \right\}$). These quantities obviously correspond roughly to the gravitational force – but what determines whether such a force exists? The answer is that gravitational acceleration depends on spatial change in the metric. For a simple example, consider gravitational time dilation in a weak field: for events at the same spatial position, there must be a separation in proper time of

$$d\tau \simeq dt \left(1 + \frac{\Delta\phi}{c^2}\right). \qquad (1.20)$$

This suggests that the gravitational acceleration should be obtained via

$$\mathbf{a} = -\frac{c^2}{2}\boldsymbol{\nabla} g_{00}. \tag{1.21}$$

More generally, we can differentiate the equation for $g_{\mu\nu}$ to get

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} + \Gamma^\beta_{\lambda\nu} g_{\beta\mu}. \tag{1.22}$$

Using the symmetry of the Γ's in their lower indices, and defining $g^{\mu\nu}$ to be the matrix inverse to $g_{\mu\nu}$, we can find an equation for the Γ's directly in terms of the metric tensor:

$$\boxed{\Gamma^\alpha_{\lambda\mu} = \tfrac{1}{2} g^{\alpha\nu}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right).} \tag{1.23}$$

Thus, the metric tensor is the crucial object in general relativity: given it, we know both the structure of spacetime and how particles will move.

1.3 Tensors and relativity

Before proceeding further, the above rather intuitive treatment should be set on a slightly firmer mathematical foundation. There are a variety of possible approaches one can take, which differ sufficiently that general relativity texts for physicists and mathematicians sometimes scarcely seem to refer to the same subject. For now, we stick with a rather old-fashioned approach, which has the virtue that it is likely to be familiar. Amends will be made later.

COVARIANT AND CONTRAVARIANT COMPONENTS So far, tensors have been met in their role as quantities that provide generally valid relations between different 4-vectors. If such relations are to be physically useful, they must apply in different frames of reference, and so the components of tensors have to change to compensate for the fact that the components of 4-vectors alter under a coordinate transformation. The transformation law for tensors is obtained from that for 4-vectors. For example, consider $c^2 d\tau^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$: substitute for dx^μ in terms of dx'^α and require that the resulting equation must have the form $c^2 d\tau^2 = g'_{\alpha\beta}\, dx'^\alpha dx'^\beta$. We then deduce the tensor transformation law

$$\boxed{g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\, g_{\mu\nu},} \tag{1.24}$$

of which law our above definition of $g_{\mu\nu}$ in terms of $\eta_{\alpha\beta}$ is an example.

Note that this transformation law looks rather like a generalization of that for a single 4-vector (with one transformation coefficient per index), but with the important difference that the coefficients are upside down in the tensor relation. For Cartesian coordinates, this would make no difference:

$$\frac{\partial x^\mu}{\partial x'^\alpha} = \frac{\partial x'^\alpha}{\partial x^\mu} = \cos\theta, \tag{1.25}$$

where θ is the angle of rotation between the two coordinate axes. In general, though,