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The classical electromagnetic field in the absence of sources

Introduction. The principal aim of this chapter is to familiarize the reader with the notation adopted in the text, as well as to introduce some concepts, such as the energy-momentum tensor of the electromagnetic field, the partition of its total angular momentum into an orbital and a spin contribution and its expansion in vector spherical harmonics, which are not usually included in an undergraduate course on electrodynamics. The chapter is entirely dedicated to the classical electromagnetic field in the absence of charges and currents. In the first two sections we present Maxwell's equations, the vector potential and different forms of the Lagrangian density of the free field from which Maxwell's equations can be obtained as Euler-Lagrange equations. In Section 1.3 we discuss briefly the properties of the field under pure Lorentz transformation and tensor notation. Then we introduce the concept of local gauge invariance and of gauge transformation, and we define the constraints leading to the Lorentz and to the Coulomb gauge. Using a canonical formalism, in Section 1.5 we obtain the Hamiltonian density of the field in the Coulomb gauge. The energy-momentum tensor of the field, the momentum and the angular momentum, along with their important conservation properties, are discussed in Section 1.6. The attention is focused on the angular momentum in Section 1.7, with a discussion of the partition into orbital and spin contributions. The mathematical properties of a general vector field are described in Section 1.8, with particular reference to the partition into longitudinal and transverse fields and to the definition of the longitudinal and transverse δ-functions and of their Fourier transform. These mathematical tools are used in Section 1.9 where the properties of the vector potential and its changes under gauge transformations are succinctly summarized. In the next section the solutions of Maxwell's equation, in the absence of charges and currents, are expanded in plane and spherical waves. This gives the opportunity for a short discussion of boundary conditions imposed on the electromagnetic field on surfaces of different shape, which is done in Section 1.11. The chapter is concluded by Section 1.12, where the plane wave field amplitude expansion, introduced in Section 1.10, is used to express various quantities relevant for the description of the dynamics of the electromagnetic field, such as the momentum, the angular momentum and the Hamiltonian density.

1.1 Maxwell's equations in the absence of sources

In the absence of charges and currents and in empty space Maxwell's equations, which describe the propagation of the electromagnetic field, take the form

$$\nabla\cdot\mathbf{E}=0\;;\nabla\times\mathbf{H}-\frac{1}{c}\dot{\mathbf{E}}=0\;;\;\nabla\cdot\mathbf{H}=0\;;\;\nabla\times\mathbf{E}+\frac{1}{c}\dot{\mathbf{H}}=0 \qquad (1.1)$$

In these equations **E** and **H** are the electric and magnetic field respectively, both functions of space coordinates $\mathbf{r}=(x_1,x_2,x_3)$ and time t, and c is the speed of light in vacuo. The Gaussian system of units is used throughout this book. Conversion tables between Gaussian and SI are given in Appendix E. It should be noted that in the present book no use is made of the field auxiliary to the magnetic field. Thus the magnetic field has been denoted by **H** rather than **B**. This choice is similar to that of Heitler (1960) and of Landau and Lifshitz (1975). On the other hand, Power (1964) and Craig and Thirunamachandran (1984) denote the magnetic field by **B**.

As is well known and as we will discuss briefly in Section 1.3, the properties of the electromagnetic field under a coordinate transformation are most conveniently expressed by the mathematical properties of the field tensor

$$F_{\mu\nu}=\begin{pmatrix} 0 & H_3 & -H_2 & -iE_1 \\ -H_3 & 0 & H_1 & -iE_2 \\ H_2 & -H_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \qquad (1.2)$$

In terms of $F_{\mu\nu}$, the first and the second pair of Equations (1.1) take either of the two particularly compact forms

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu}=0\;;\;e_{\kappa\lambda\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x_\lambda}=0\;\;\text{or}\;\;\frac{\partial F_{\lambda\mu}}{\partial x_\nu}+\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{F_{\nu\lambda}}{\partial x_\mu}=0 \qquad (1.3)$$

where $e_{\kappa\lambda\mu\nu}$ is the completely antisymmetric four-tensor whose components change sign under the interchange of any pair of indices and which has $e_{1234} = 1$ (see e.g. Landau and Lifshitz 1975). In (1.3) the convention of implicit summation over repeated indices is used, and the four-vector x_μ is defined as (x_1, x_2, x_3, x_4) with $x_4 = ict$. Since we shall limit our discussion to the domain of the special theory of relativity, no distinction is necessary between covariant and contravariant components and no metric tensor is introduced (see Sakurai 1982). Note also that all Greek indices run from 1 to 4; in the future we shall also use Latin indices which run from 1 to 3. It is evident that form (1.3) of Maxwell's equations is covariant under Lorentz transformations.

Another form of Maxwell's equations can be obtained in terms of the four-vector A_μ, which is related to the field tensor by (see e.g. Sakurai 1982)

$$\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = F_{\mu\nu} \tag{1.4}$$

From (1.2) we obtain

$$\nabla \times \mathbf{A} = \mathbf{H} \; ; \; \nabla A_4 = -i(\mathbf{E} + \frac{1}{c}\dot{\mathbf{A}}) \tag{1.5}$$

Clearly A_i are the components of the vector potential whereas A_4 coincides with iV, where V is the scalar potential familiar from elementary electromagnetism (see e.g. Bleaney and Bleaney 1985). Substituting (1.5) into the second pair of (1.1) yields two trivial identities. The first two of (1.1), however, yield

$$\nabla^2\mathbf{A} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\dot{V}\right) - \frac{1}{c^2}\ddot{\mathbf{A}} = 0 \; ; \; \nabla^2 V + \frac{1}{c}\nabla \cdot \dot{\mathbf{A}} = 0 \tag{1.6}$$

or in tensor notation

$$\frac{\partial^2 A_\mu}{\partial x_\nu^2} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu}{\partial x_\nu}\right) = 0 \tag{1.7}$$

Equations (1.6) or, equivalently, (1.7) are the third form of Maxwell's equations *in vacuo* considered here. The slight abuse of language should be noted. Strictly speaking in fact (1.7) are the equations of field A_μ rather than Maxwell's equations which are normally expressed in terms of **E** and **H**.

1.2 Lagrangian of the free field

The equations of motion of any field should be derivable from an appropriate Lagrangian density using the Euler-Lagrange equations. It is easy to show that the following Lagrangian density

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{8\pi}(\mathbf{H}^2 - \mathbf{E}^2) = -\frac{1}{16\pi}F_{\mu\nu}F_{\mu\nu} \\ &= -\frac{1}{16\pi}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right)\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right)\end{aligned} \tag{1.8}$$

is suitable for our needs. It should be noted that in a canonical formalism the Lagrangian density of any field should be expressed in terms of the field amplitude and of its derivatives, whereas its Hamiltonian density should be expressed in terms of the field amplitudes and of their canonically conjugate momenta. In this book we shall occasionally infringe this rule when extreme precision of language is superfluous. In (1.8), for example, the first two forms of $\mathcal{L}_0$ should not, strictly speaking, be considered as Lagrangian density, although they do coincide with the value taken by the Lagrangian density when expressed in terms of **E** and **H**. It is evident, however, that no confusion should arise in the case at hand, where it is natural to adopt the third form of (1.8) to obtain the Euler-Lagrange equations. This form for $\mathcal{L}_0$ in (1.8) is a function of the first derivatives of A_μ only, and not of A_μ. Therefore the Euler-Lagrange equations, valid for a general Lagrangian density $\mathcal{L}$ function of $\frac{\partial A_\mu}{\partial x_\nu}$ and of A_μ,

$$\frac{\partial}{\partial x_\nu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_\mu}{\partial x_\nu}\right)} - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \tag{1.9}$$

reduce for $\mathcal{L}_0$ to

$$\frac{\partial}{\partial x_\nu}\frac{\partial \mathcal{L}_0}{\partial\left(\frac{\partial A_\mu}{\partial x_\nu}\right)} = 0 \tag{1.10}$$

Moreover, from (1.8)

$$\begin{aligned}\frac{\partial \mathcal{L}_0}{\partial\left(\frac{\partial A_\mu}{\partial x_\nu}\right)} &= -\frac{1}{16\pi}\left\{\frac{\partial}{\partial\left(\frac{\partial A_\mu}{\partial x_\nu}\right)}\left[\left(\frac{\partial A_\sigma}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_\sigma}\right)\left(\frac{\partial A_\sigma}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_\sigma}\right)\right]\right\} \\ &= -\frac{1}{8\pi}\left[\frac{\partial}{\partial\left(\frac{\partial A_\mu}{\partial x_\nu}\right)}\left(\frac{\partial A_\sigma}{\partial x_\lambda}\frac{\partial A_\sigma}{\partial x_\lambda} - \frac{\partial A_\sigma}{\partial x_\lambda}\frac{\partial A_\lambda}{\partial x_\sigma}\right)\right] \\ &= \frac{1}{4\pi}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right)\end{aligned} \tag{1.11}$$

Substitution of result (1.11) into the Euler-Lagrange equation (1.10) immediately yields Maxwell's equations in the form (1.7).

It should be remarked that $\mathcal{L}_0$ is not the only Lagrangian density giving rise to Maxwell's equations. For example, if we add to $\mathcal{L}_0$ a scalar which is the four-divergence of a four-vector Γ_μ, function of A_μ and x_μ, the equations of motion for the new Lagrangian density $\mathcal{L}_0 + \frac{\partial \Gamma_\mu}{\partial x_\mu}$ will coincide with the equations of motion for $\mathcal{L}_0$. This is due to the fact that the Euler-Lagrange equations are obtained by varying the action $\int \mathcal{L} dx$, and the four-divergence term introduced above contributes a vanishing boundary term to the variation of the action, as discussed by Barut (1980). In fact, more generally but for the same reason, two Lagrangian densities which differ by a term which vanishes upon space-time integration and possibly upon application of additional appropriate constraints, yield the same Euler-Lagrange equations (Bogoliubov and Shirkov (1960). Examples of such equivalent Lagrangian densities are (see e.g. Schweber 1964)

$$\mathcal{L}_1 = \mathcal{L}_0 - \frac{1}{8\pi}\left(\frac{\partial A_\mu}{\partial x_\mu}\right)^2 ; \quad \mathcal{L}_2 = -\frac{1}{8\pi}\left(\frac{\partial A_\mu}{\partial x_\nu}\right)^2 \tag{1.12}$$

1.3 Pure Lorentz transformations

Any four-vector, such as x_μ, under a homogeneous Lorentz transformation yields a new four vector such that

$$x'_\mu = \Lambda_{\mu\nu} x_\nu \tag{1.13}$$

where $\Lambda_{\mu\nu}$ are matrix elements of a 4×4 matrix Λ representing the effects of the Lorentz transformation (see e.g. Ugarov 1982). The four-vector transformation property specified by (1.13) is called "Lorentz covariance". If Λ is a pure Lorentz transformation (Goldstein 1980) the matrix elements are given by

$$\begin{aligned} &\Lambda_{ij} = \delta_{ij} + \beta^{-2}\beta_i\beta_j(\gamma - 1) ; \quad \Lambda_{i4} = i\beta_i\gamma ; \\ &\Lambda_{4j} = -i\beta_j\gamma ; \quad \Lambda_{44} = \gamma \end{aligned} \tag{1.14}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$, $\gamma = (1 - \beta^2)^{-1/2}$ and $\mathbf{v}$ is the relative velocity of the transformation. Lorentz transformations are orthogonal and their matrix elements satisfy

$$\Lambda_{\mu\nu}\Lambda_{\mu\lambda} = \delta_{\nu\lambda} ; \quad \left(\Lambda^{-1}\right)_{\mu\nu} = \Lambda_{\nu\mu} \tag{1.15}$$

The latter property can be exploited to invert (1.13) as

$$x_\mu = \left(\Lambda^{-1}\right)_{\mu\nu} x'_\nu = \Lambda_{\nu\mu} x'_\nu \tag{1.16}$$

Consequently from (1.16)

$$\frac{\partial}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu} \frac{\partial}{\partial x_\nu} = \Lambda_{\mu\nu} \frac{\partial}{\partial x_\nu} \tag{1.17}$$

which shows that the operator $\frac{\partial}{\partial x_\nu}$, sometimes called the four-gradient, is also a four-vector.

The scalar product of two four-vectors, such as x_μ and y_μ, is invariant under a Lorentz transformation, in view of (1.15). In fact

$$x'_\mu y'_\mu = \Lambda_{\mu\nu} x_\nu \Lambda_{\mu\lambda} y_\lambda = \delta_{\nu\lambda} x_\nu y_\lambda = x_\mu y_\mu \tag{1.18}$$

A quantity which is invariant under a Lorentz transformation is called a "Lorentz scalar". A tensor with two indices, such as $F_{\mu\nu}$, is a "tensor of second rank". A tensor of second rank transforms according to the rule

$$F'_{\mu\nu} = \Lambda_{\mu\lambda} \Lambda_{\nu\rho} F_{\lambda\rho} \tag{1.19}$$

Hence the quantity $F_{\mu\nu} F_{\mu\nu}$ transforms as

$$F'_{\mu\nu} F'_{\mu\nu} = \Lambda_{\mu\lambda} \Lambda_{\nu\rho} F_{\lambda\rho} \Lambda_{\mu\sigma} \Lambda_{\nu\tau} F_{\sigma\tau} = \delta_{\lambda\sigma} \delta_{\rho\tau} F_{\lambda\rho} F_{\sigma\tau} = F_{\mu\nu} F_{\mu\nu} \tag{1.20}$$

and thus is a Lorentz scalar.

Thus the three Lagrangian densities $\mathcal{L}_i$ appearing in (1.8) and (1.12) are Lorentz scalars and invariant under Lorentz transformations. It is also easy to check that Maxwell's equations are Lorentz-covariant, as it is particularly evident when they are expressed in the form (1.3). This means that they are of the same form in reference frames related by Lorentz transformations, in accord with the principles of special relativity. Indeed, it is possible to show in general that the Lagrangian density of any field must be a Lorentz scalar if the Euler-Lagrange equations are to be Lorentz-covariant.

1.4 Gauge transformations

Suppose we have obtained a four-vector A_μ satisfying relation (1.4) for a given field tensor $F_{\mu\nu}$. We now add to A_μ the four-gradient of a scalar field $\chi(x_\mu)$, a function of position and time, and obtain a new four-vector

$A_\mu + \partial\chi/\partial x_\mu$. It is easy to see that this new four-vector, like A_μ, satisfies (1.4), for the same field tensor $F_{\mu\nu}$. In fact

$$\frac{\partial A_\nu}{\partial x_\mu} + \frac{\partial^2 \chi}{\partial x_\mu x_\nu} - \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial^2 \chi}{\partial x_\nu x_\mu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = F_{\mu\nu} \tag{1.21}$$

Indeed, until the scalar $\chi(\mathbf{r}, t)$ is subjected to some condition which determines it, there are an infinite number of four-vectors $A_\mu + \partial\chi/\partial x_\mu$ which correspond to the same electromagnetic field $F_{\mu\nu}$. Clearly the Lagrangian density $\mathcal{L}_0$ given by (1.8) is invariant with respect to the transformation A_μ to $A_\mu + \partial\chi/\partial x_\mu$. Such an invariance is called local gauge invariance, or gauge invariance of the second kind, where χ is a function of the space-time point x_μ. Gauge invariance of the second kind is discussed by Doughty (1990) and Mandl and Shaw (1984) among others. On the other hand $\mathcal{L}_1$ and $\mathcal{L}_2$ in (1.12) are not gauge invariant.

As for Maxwell's equations, clearly forms (1.1) and (1.3) are gauge invariant, but form (1.6) in terms of the four-potential, or equivalently (1.7), is not. Clearly the four-potential A_μ cannot be regarded as representing a physical observable, because the form of the equation of motion of such an observable cannot depend on an arbitrary mathematical function such as $\partial\chi/\partial x_\mu$. It should be emphasized, however, that once A_μ has been chosen according to some criteria, the electromagnetic field is uniquely determined on the basis of (1.4). The choice of these criteria is a procedure which is called "fixing the gauge". This absence of a unique correspondence between the description in terms of **E** and **H** and the description in terms of A_μ is to be expected on the basis of the fact that Maxwell's equations (1.1) are first-order differential equations, whose solution necessitates knowledge of six functions of **r** as initial conditions. These are the three components of **E** and **H** for each point of space. On the other hand, Maxwell's equations (1.6) are of second order, and their solution implies knowledge of eight functions, which are the components A_μ and their time derivatives. Thus the A_μ description of the field displays some redundancy of dynamical variables, which calls for the introduction of constraints between some of them. This point is discussed in clear terms by Cohen-Tannoudji *et al.* (1989). These constraints are the conditions which fix the gauge.

Consider, for example, the following constraint

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{A} + \frac{1}{c}\dot{V} = 0 \tag{1.22}$$

which fixes a gauge called the Lorentz gauge. The scalar nature of the quantity on the LHS of (1.22), which is a four-divergence and Lorentz invariant, makes this gauge particularly useful for treatments which fully exploit the relativistic nature of electrodynamics. Then the second term on the LHS of (1.7) vanishes, and Maxwell's equations (1.6) take the form

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} = 0\ ;\ \nabla^2 V - \frac{1}{c^2}\ddot{V} = 0 \tag{1.23}$$

It is interesting to note that constraint (1.22) does not completely lead to a unique relation between A_μ and $(\mathbf{E},\mathbf{H})$. In fact, we see that any new four-vector $A_\mu + \partial\chi/\partial x_\mu$, where A_μ satisfies (1.22) and $\chi(\mathbf{r},t)$ is a scalar subject to the condition

$$\nabla^2\chi - \frac{1}{c^2}\ddot{\chi} = 0 \tag{1.24}$$

satisfies the same constraint (1.22). This shows that the Lorentz gauge is really a class of subgauges determined by the Lorentz condition. The arbitrariness of χ can be exploited to eliminate V from Maxwell's equation, by choosing among the various χ solutions of (1.24) that particular χ for which $V + \dot{\chi}/c = 0$ (see e.g. Heitler 1960). In this case the Lorentz constraint (1.22) reduces to

$$\nabla\cdot\mathbf{A} = 0\ ,\ \ V = 0 \tag{1.25}$$

and Maxwell's equations are simply given by the first of (1.23).

Another gauge, the so-called "transverse" or "Coulomb" gauge, is obtained by imposing a constraint only on the first three components of A_μ, in the form

$$\nabla\cdot\mathbf{A} = 0 \tag{1.26}$$

and not on A_4. Thus the Coulomb gauge (1.26) should not be confused with the Lorentz subgauge (1.25) where V is constrained to vanish. In the absence of charges and currents and in infinite unbounded space, however, these two gauges coincide, since substitution of (1.26) in (1.6) yields

$$\nabla^2\mathbf{A} - \frac{1}{c}\nabla\dot{V} - \frac{1}{c^2}\ddot{\mathbf{A}} = 0\ ;\ \nabla^2 V = 0 \tag{1.27}$$

and since the only nondiverging solution of Laplace's equation for V, which vanishes at infinity, is the null $V = 0$ solution. This point is discussed e.g. by Morse and Feshbach (1953).

1.5 Hamiltonian density of the free field in the Coulomb gauge

The usual Hamiltonian formalism can be applied to the free electromagnetic field. We start from the gauge invariant Lagrangian density $\mathcal{L}_0$ in (1.8). With A_μ as the generalized coordinates of the field one should be able to obtain conjugate momenta Π_μ

$$\Pi_\mu = \frac{1}{ic}\frac{\partial \mathcal{L}_0}{\partial\left(\frac{\partial A_\mu}{\partial x_4}\right)} \tag{1.28}$$

and the Hamiltonian density as

$$\mathcal{H}_0 = ic\Pi_\mu \frac{\partial A_\mu}{\partial x_4} - \mathcal{L}_0 \tag{1.29}$$

Such a general approach, however, is not as simple as it looks. In fact, from (1.11) we immediately see that Π_4, the momentum conjugate to the scalar potential V, vanishes identically. This is going to cause some difficulties, for example in connection with quantization where one is expected to impose noncommutation of Π_4 and V. The relativistic field theorist's way out of this difficulty is to change the Lagrangian while keeping the four generalized coordinates A_μ for the field. This permits us to preserve the manifest relativistic features of the theory at each step of the calculations, but it gives rise to a series of formal difficulties which are unnecessary for our purposes, as discussed by Schweber (1964).

We shall take a different approach, and exploit the gauge invariance of $\mathcal{L}_0$ in order to reduce the number of field coordinates. This will give rise to a theory which is not manifestly relativistically covariant, but it is much more simple formally. Thus, remembering we are discussing the case of no charges and currents, we choose the Lorentz subgauge with constraints (1.25), or equivalently the Coulomb gauge, and set $A_4 = 0$. Then (1.8) becomes (Craig and Thirunamachandran 1984)

$$\mathcal{L}_0 = \frac{1}{8\pi}\left\{\frac{1}{c^2}\dot{\mathbf{A}}^2 - \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)^2\right\} \tag{1.30}$$

Expression (1.30) depends on three generalized coordinates only, since A_4 does not appear. This coordinate will reappear, however, if we subject $\mathcal{L}_0$ to a pure Lorentz transformation, thereby spoiling manifest Lorentz invariance. This means that our approach is not really suited to describing processes which involve Lorentz transformations where non-Galilean features are important. On the other hand the formalism should handle low-energy processes properly, which are those of interest for the present book. This point is discussed by Cohen-Tannoudji *et al.* (1989).

The equations of motion of the field are now given by the first of (1.23) alone or, equivalently, by the first of (1.27) with $V = 0$. The momentum components conjugate to A_i are, from (1.28)

$$\Pi_i = \frac{\partial \mathcal{L}_0}{\partial \dot{A}_i} = \frac{1}{4\pi c^2}\dot{A}_i = -\frac{1}{4\pi c}E_i \tag{1.31}$$

where (1.5) with $A_4 = 0$ has been used. The Hamiltonian density is obtained from (1.29) and

$$\begin{aligned}\mathcal{H}_0 &= \Pi_i \frac{\partial A_i}{\partial t} - \mathcal{L}_0 \\ &= \frac{1}{8\pi}\left\{\frac{1}{c^2}\dot{A}_i^2 + \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)^2\right\} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2)\end{aligned} \tag{1.32}$$

1.6 Energy-momentum tensor and conservation laws

Following Heitler's notation (1960), we define the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\lambda}F_{\lambda\nu} + \frac{1}{4}\delta_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho}\right] \tag{1.33}$$

From (1.2) it is easy to obtain the following expression for the elements of $T_{\mu\nu}$

$$\begin{aligned}&T_{ii} = \frac{1}{4\pi}\left(E_i^2 + H_i^2\right) - \mathcal{H}_0;\ T_{44} = \frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{H}^2\right) \equiv \mathcal{H}_0;\\ &T_{ij} = T_{ji} = \frac{1}{4\pi}\left(E_iE_j + H_iH_j\right) \quad (i \neq j);\\ &T_{i4} = T_{4i} = -\frac{i}{4\pi}(\mathbf{E}\times\mathbf{H})_i\end{aligned} \tag{1.34}$$

where $\mathcal{H}_0$ is the energy density. T_{44} coincides with the Hamiltonian density (1.32) which was obtained in the Coulomb gauge. So the Hamiltonian density, contrary to the Lagrangian density of the free field, is not a Lorentz scalar, but transforms like the time-time component of a tensor of second rank. We remark that $T_{\mu\nu}$ in (1.33) is gauge invariant, symmetric ($T_{\mu\nu} = T_{\nu\mu}$) and traceless ($T_{\mu\mu} = 0$). We also remark that the three components T_{i4} are related to the momentum density of the field,