POTENTIAL THEORY IN GRAVITY AND MAGNETIC APPLICATIONS

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Introduction

> Though this be madness, yet there is method in't.
>
> *(William Shakespeare)*

> I think I did pretty well, considering I started out with nothing but a bunch of blank paper.
>
> *(Steve Martin)*

Pierre Simon, Marquis de Laplace, showed in 1782 that Newtonian potential obeys a simple differential equation. Laplace's equation, as it now is called, arguably has become the most universal differential equation in the physical sciences because of the wide range of phenomena that it describes. The theory of the potential spawned by Laplace's equation is the subject of this book, but with particular emphasis on the application of this theory to gravity and magnetic fields of the earth and in the context of geologic and geophysical investigations.

A Brief History of Magnetic and Gravity Methods

The geomagnetic field must surely rank as the longest studied of all the geophysical properties of the earth. Curiosity about the mutual attraction of lodestones can be traced back at least to the time of Thales, a philosopher of ancient Greece in the sixth century B.C. (Needham [194]). The tendency of lodestones to align preferentially in certain directions was known in China by the first century A.D., and perhaps as early as the second century B.C. This apparently was the first recognition that the earth is associated with a property that affects magnetic objects, thus paving the way for the advent of the magnetic compass in China and observations of magnetic declination.

The compass arrived in Europe much later, probably late in the twelfth century A.D., but significant discoveries were to follow. Petrus Peregrinus, a scholar of thirteenth-century Italy, performed several important experiments on spherical pieces of lodestone. His findings, written in 1269, described for the first time the concepts of magnetic polarity, magnetic meridians, and the idea that like poles repel but opposite poles attract. Georg Hartmann, Vicar of Nuremberg, was the first European to measure magnetic declination in about 1510. He also discovered magnetic inclination in 1544, but his writings went undiscovered until after Robert Norman, an English hydrographer, published his own careful experiments on inclination conducted in 1576. In 1600, William Gilbert, physician to Queen Elizabeth I, published his landmark treatise, *De Magnete*, culminating centuries of European and Chinese thought and experimentation on the geomagnetic field. Noting that the earth's magnetic field has a form much like that of a spherically shaped piece of lodestone, Gilbert proclaimed that *"magnus magnes ipse est globus terrestris"* ("the whole earth is a magnet"), and magnetism thus became the first physical property, other than roundness, attributed to the earth as a whole (Merrill and McElhinny [183]). In 1838, the German mathematician Carl Friederich Gauss gave geomagnetic observations their first global-scale mathematical formalism by applying spherical harmonic analysis to a systematic set of magnetic measurements available at the time.

The application of magnetic methods to geologic problems advanced in parallel with the development of magnetometers. Geologic applications began at least as early as 1630, when a sundial compass was used to prospect for iron ore in Sweden (Hanna [110]), thus making magnetic-field interpretation one of the oldest of the geophysical exploration techniques. Early measurements of the magnetic field for exploration purposes were made with land-based, balanced magnets similar in principle of operation to today's widely used gravity meters. Max Thomas Edelmann used such a device during the first decade of this century to make the first airborne magnetic measurements via balloon (Heiland [121]). It was soon recognized that measurements of the magnetic field via aircraft could provide superior uniform coverage compared to surface measurements because of the aircraft's ability to quickly cover remote and inaccessible areas, but balanced-magnet instruments were not generally amenable to the accelerations associated with moving platforms. It was military considerations, related to World War II, that spurred the development of a suitable magnetometer for routine aeromagnetic measurements. In 1941, Victor Vacquier, Gary Muffly, and R. D. Wyckoff, employees of Gulf Research and Development Company under contract with the U.S. government, modified 10-year-old flux-gate technology, combined it with suitable stabilizing equipment, and thereby developed a magnetometer for airborne detection of submarines. In 1944, James R. Balsley and Homer Jensen of the U.S. Geological Survey used a magnetometer of similar design in the first modern airborne geophysical survey near Boyertown, Pennsylvania (Jensen [143]).

A second major advance in magnetometer design was the development of the proton-precession magnetometer by Varian Associates in 1955. This relatively simple instrument measures the magnitude of the total field without the need for elaborate stabilizing or orienting equipment. Consequently, the proton-precession magnetometer is relatively inexpensive and easy to operate and has revolutionized land-based and shipborne measurements. Various other magnetometer designs have followed with greater resolution (Reford [240]) to be sure, but the proton-precession magnetometer remains a mainstay of field surveys.

Shipborne magnetic measurements were well under way by the 1950s. By the mid 1960s, ocean-surface measurements of magnetic intensity in the Northeast Pacific (Raff and Mason [234]) had discovered curious anomalies lineated roughly north–south. Fred Vine and Drummond Matthews [286] and, independently, Lawrence Morley and Andre Larochelle [186] recognized that these lineations reflect a recording of the reversing geomagnetic field by the geologic process of seafloor spreading, and thus was spawned the plate-tectonic revolution.

The gravity method too has a formidable place in the history of science. The realization that the earth has a force of attraction surely must date back to our initial awareness that dropped objects fall to the ground, observations that first were quantified by the well-known experiments of Galileo Galilei around 1590. In 1687 Isaac Newton published his landmark treatise, *Philosophiae Naturalis Principia Mathematica*, in which he proposed (among other revolutionary concepts) that the force of gravity is a property of all matter, Earth included.

In 1672 a French scholar, Jean Richer, noted that a pendulum-based clock designed to be accurate in Paris lost a few minutes per day in Cayenne, French Guiana, and so pendulum observations were discovered as a way to measure the spatial variation of the geopotential. Newton correctly interpreted the discrepancy between these two measurements as reflecting the oblate shape of the earth. The French believed otherwise at the time, and to prove the point, the French Academy of Sciences sent two expeditions, one to the equatorial regions of Ecuador and the other to the high latitudes of Sweden, to carefully measure and compare the length of a degree of arc at both sites (Fernie [88, 89, 90]). The Ecuador expedition was led by several prominent French scientists, among them Pierre Bouguer, sometimes credited for the first careful observations of the shape of the earth and for whom the "Bouguer anomaly" is named.

The reversible pendulum was constructed by H. Kater in 1818, thereby facilitating absolute measurements of gravity. Near the end of the same century, R. Sterneck of Austria reported the first pendulum instrument and used it to measure gravity in Europe. Other types of pendulum instruments followed, including the first shipborne instrument developed by F. A. Vening Meinesz of The Netherlands in 1928, and soon gravity measurements were being recorded worldwide. The Hungarian geodesist, Roland von Eötvös, constructed the first torsional balance in 1910. Many gravity meters of various types were developed and patented during 1928 to 1930 as U.S. oil companies became interested in exploration applications. Most modern instruments suitable for field studies, such as the LaCoste and Romberg gravity meter and the Worden instrument, involve astatic principles in measuring the vertical displacement of a small mass suspended from a system of delicate springs and beams. Various models of the LaCoste and Romberg gravity meter are commonly used in land-based and shipborne studies and, more recently, in airborne surveys (e.g., Brozena and Peters [43]).

The application of gravity measurements to geological problems can be traced back to the rival hypotheses of John Pratt and George Airy published between 1855 and 1859 concerning the isostatic support of topography. They noted that plumb lines near the Himalayas were deflected from the vertical by amounts less than predicted by the topographic mass of the mountain range. Both Airy and Pratt argued that in the absence of forces other than gravity, the rigid part of the crust and mantle "floats" on a mobile, denser substratum, so the total mass in any vertical column down to some depth of compensation must balance from place to place. Elevated regions, therefore, must be compensated at depth by mass deficiencies, whereas topographic depressions are underlain by mass excesses. Pratt explained this observation in terms of lateral variations in density; that is, the Himalayas are elevated because they are less dense than surrounding crust. Airy proposed, on the other hand, that the crust has laterally uniform density but variable thickness, so mountain ranges rise above the surrounding landscape by virtue of underlying crustal roots.

The gravity method also has played a key role in exploration geophysics. Hugo V. Boeckh used an Eötvös balance to measure gravity over anticlines and domes and explained his observations in terms of the densities of rocks that form the structures. He thus was apparently the first to recognize the application of the gravity method in the exploration for petroleum (Jakosky [140]). Indeed the first oil discovered in the United States by geophysical methods was located in 1926 using gravity measurements (Jakosky [140]).

About This Book

Considering this long and august history of the gravity and magnetic methods, it might well be asked (as I certainly have done during the waning stages of this writing) why a new textbook on potential theory is needed now. I believe, however, that this book will fill a significant gap. As a graduate student at Stanford University, I quickly found myself involved in a thesis topic that required a firm foundation in potential theory. It seemed to me then, and I find it true today as a professional geophysicist, that no single textbook is available covering the topic of potential theory while emphasizing applications to geophysical problems. The classic texts on potential theory published during the middle of this century are still available today, notably those by Kellogg [146] and by Ramsey [235] (which no serious student of potential theory should be without). These books deal thoroughly with the fundamentals of potential theory, but they are not concerned particularly with geophysical applications. On the other hand, several good texts are available on the broad topics of applied geophysics (e.g., Telford, Geldart, and Sheriff [279]) and global geophysics (e.g., Stacey [270]). These books cover the wide range of geophysical methodologies, such as seismology, electromagnetism, and so forth, and typically devote a few chapters to gravity and magnetic methods; of necessity they do not delve deeply into the underlying theory.

This book attempts to fill the gap by first exploring the principles of potential theory and then applying the theory to problems of crustal and lithospheric geophysics. I have attempted to do this by structuring the book into essentially two parts. The first six chapters build the foundations of potential theory, relying heavily on Kellogg [146], Ramsey [235], and Chapman and Bartels [56]. Chapters 1 and 2 define the meaning of a potential and the consequences of Laplace's equation. Special attention is given therein to the all-important Green's identities, Green's functions, and Helmholtz theorem. Chapter 3 focuses these theoretical principles on Newtonian potential, that is, the gravitational potential of mass distributions in both two and three dimensions. Chapters 4 and 5 expand these discussions to magnetic fields caused by distributions of magnetic media. Chapter 6 then formulates the theory on a spherical surface, a topic of obvious importance to global representations of the earth's gravity and magnetic fields.

The last six chapters apply the foregoing principles of potential theory to gravity and magnetic studies of the crust and lithosphere. Chapters 7 and 8 examine the gravity and magnetic fields of the earth on a global and regional scale and describe the calculations and underlying theory by which measurements are transformed into "anomalies." These discussions set the stage for the remaining chapters, which provide a sampling of the myriad schemes in the literature for interpreting gravity and magnetic anomalies. These schemes are divided into the forward method (Chapter 9), the inverse method (Chapter 10), inverse and forward manipulations in the Fourier domain (Chapter 11), and methods of data enhancement (Chapter 12). Here I have concentrated on the mathematical rather than the technical side of the methodology, neglecting such topics as the nuts-and-bolts operations of gravity meters and magnetometers and the proper strategies in designing gravity or magnetic surveys.

Some of the methods discussed in Chapters 9 through 12 are accompanied by computer subroutines in Appendix B. I am responsible for the programming therein (user beware), but the methodologies behind the algorithms are from the literature. They include some of the "classic" techniques, such as the so-called Talwani method discussed in Chapter 9, and several more modern methods, such as the horizontal-gradient calculation first discussed by Cordell [66]. Those readers wishing to make use of these subroutines should remember that the programming is designed to instruct rather than to be particularly efficient or "elegant."

It would be quite beyond the scope of this or any other text to fully describe all of the methodologies published in the modern geophysical literature. During 1992 alone, *Geophysics* (the technical journal of the U.S.-based Society of Exploration Geophysicists) published 17 papers that arguably should have been covered in Chapters 9 through 12. Multiply that number by the several dozen international journals of similar stature and then times the 50 some-odd years that the modern methodology has been actively discussed in the literature, and it becomes clear that each technique could not be given its due. Instead, my approach has been to describe the various methodologies with key examples from the literature, including both classic algorithms and promising new techniques, and with apologies to all of my colleagues not sufficiently cited!

Acknowledgments

The seeds of this book began in graduate-level classes that I prepared and taught at Oregon State University and Stanford University between 1973 and 1990. The final scope of the book, however, is partly a reflection of interactions and discussions with many friends and colleagues. Foremost are my former professors at Stanford University during my graduate studies, especially Allan Cox, George Thompson, and Jon Claerbout, who introduced me to geological applications of potential theory and time-series analysis. My colleagues at the U.S. Geological Survey, Stanford University, Oregon State University, and elsewhere were always available for discussions and fomentation, especially Robert Jachens, Robert Simpson, Thomas Hildenbrand, Richard Saltus, Andrew Griscom, V. J. S. Grauch, Gerald Connard, Gordon Ness, and Michael McWilliams. I am grateful to Richard Saltus and Gregory Schreiber for carefully checking and critiquing all chapters, and to William Hinze, Tiki Ravat, Robert Langel, and Robert Jachens for reviewing and proofreading various parts of early versions of this manuscript. I am especially grateful to Lauren Cowles, my chief contact and editor at Cambridge University Press, for her patience, assistance, and flexible deadlines.

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Richard J. Blakely