PART I

VECTOR AND TENSOR ALGEBRA

Throughout this book:

(i) *Lightface* Latin and Greek letters generally denote *scalars*.
(ii) *Boldface lowercase* Latin and Greek letters generally denote *vectors*, but the letters $o$, $x$, $y$, and $z$ are reserved for *points*.
(iii) *Boldface uppercase* Latin and Greek letters generally denote *tensors*, but the letters $X$, $Y$, and $Z$ are reserved for *points*.
1 Vector Algebra

We assume that the reader has had a basic course in vector algebra and, therefore, in introducing this subject, we take a relaxed approach that does not begin with formal definitions of point and vector spaces.

Roughly speaking, a point \( x \) is a dot in space and a vector \( v \) is an arrow that may be placed anywhere in space. As an example from everyday life, on a street map of a city

\[
x \overset{\text{def}}{=} \text{the corner of 4th Street and 5th Avenue}
\]

might describe a point on the map, and to describe a second point \( y \) one-quarter of a mile northeast of \( x \) one might let

\[
v \overset{\text{def}}{=} \text{the vector whose direction is northeast and whose length is one-quarter mile and write } y = x + v.
\]

Thus, it would seem that a reasonable definition of a point space would require two basic notions: that of a point and that of a vector. Of course, granted a choice of origin, one can identify all points with their vectors from \( o \); but such an identification is artificial, since there is no intrinsic way of defining \( o \). (What point on a street map would you call the origin?)

The space under consideration will always be a three-dimensional Euclidean point space \( \mathcal{E} \). The term point will be reserved for elements of \( \mathcal{E} \) and the term vector for elements of the associated vector space \( \mathcal{V} \). Then:

(i) The difference \( v = y - x \) between the points \( y \) and \( x \) is a vector.

(ii) The sum \( y = x + v \) of a point \( x \) and a vector \( v \) is a point.

(iii) Unlike the sum of two vectors, the sum of two points has no meaning.

1.1 Inner Product. Cross Product

Our assumption that the point space \( \mathcal{E} \) be Euclidean automatically endows the associated vector space \( \mathcal{V} \) with an inner product.\(^3\) We use the standard notation of vector analysis. In particular,

- The **inner product** (a scalar) and **cross product** (a vector)\(^4\) of vectors \( u \) and \( v \) are respectively designated by

\[
u \cdot v \quad \text{and} \quad u \times v.
\]

\(^3\) The inner product is often referred to as the dot product.

\(^4\) We assume that the reader has some familiarity with these notions. The cross product is ordered in the sense that the cross product \( u \times v \) of \( u \) and \( v \) is not generally equal to the cross product \( v \times u \) of \( v \) and \( u \).
4 Vector Algebra

Figure 1.1. The parallelogram \( \mathcal{P} \) defined by the vectors \( \mathbf{u} \) and \( \mathbf{v} \) and the direction of \( \mathbf{u} \times \mathbf{v} \) determined by the right-hand screw rule.

The inner product determines the magnitude (or length) of a vector \( \mathbf{u} \) via the relation

\[
|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}},
\]

and the angle

\[
\theta = \angle(\mathbf{u}, \mathbf{v})
\]

between nonzero vectors \( \mathbf{u} \) and \( \mathbf{v} \) is defined by

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \quad (0 \leq \theta \leq \pi).
\]

Since \(-|\mathbf{u}| |\mathbf{v}| \leq \mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u}| |\mathbf{v}|\), this definition assigns exactly one angle \( \theta \) to each pair of nonzero vectors \( \mathbf{u} \) and \( \mathbf{v} \). Trivially,

\[
\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta;
\]

this relation is often used to define the inner product.

With regard to the cross product, the magnitude

\[
|\mathbf{u} \times \mathbf{v}|
\]

represents the area spanned by the vectors \( \mathbf{u} \) and \( \mathbf{v} \); that is, the area of the parallelogram \( \mathcal{P} \) defined by these vectors as indicated in Figure 1.1; this area is nonzero if and only if \( \mathbf{u} \) and \( \mathbf{v} \) are linearly independent. Further, and what is most important, if \( \mathbf{u} \) and \( \mathbf{v} \) are linearly independent, then:

(i) the magnitude of \( \mathbf{u} \times \mathbf{v} \) is given by

\[
|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta \quad (0 < \theta < \pi);
\]

(ii) the vector \( \mathbf{u} \times \mathbf{v} \) is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \) with direction given by the right-hand screw rule.\(^6\)

\(^5\) Vectors \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) are linearly dependent if, for some choice of the scalars \( a \), \( b \), and \( c \), not all zero,

\[
a \mathbf{u} + b \mathbf{v} + c \mathbf{w} = \mathbf{0}.
\]

A similar definition applies to a pair of vectors \( \mathbf{u} \) and \( \mathbf{v} \). Finally, a set of vectors is linearly independent if it is not linearly dependent.

\(^6\) Asserting formally that a right-hand screw revolver from \( \mathbf{u} \) to \( \mathbf{v} \) will advance in its nut toward \( \mathbf{u} \times \mathbf{v} \) (Figure 1.1); cf. Brand (1947, §16) and Jeffrey (2002, §2.3). A simple consequence of this is that \( \mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v} \) and thus, in particular, that the cross product of \( \mathbf{u} \) and \( \mathbf{v} \) vanishes whenever \( \mathbf{u} \) and \( \mathbf{v} \) are linearly dependent (Footnotes 4 and 5).
1.1 Inner Product. Cross Product

Since $\mathbf{u} \times \mathbf{v}$ gives the area of $\mathcal{P}$, while $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathcal{P}$, $\mathbf{u} \times \mathbf{v}$ is sometimes referred to as the area vector of $\mathcal{P}$.

Similarly, 

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \quad (1.4)$$

represents the volume spanned by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$; that is, the volume of the parallelepiped defined by these vectors as indicated in Figure 1.2. If this volume is nonzero, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, then the triad $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis for $\mathcal{V}$ in the sense that any vector $\mathbf{a}$ may be uniquely represented in terms of that triad, that is, there are unique scalars $\alpha$, $\beta$, and $\gamma$ such that

$$\mathbf{a} = \alpha \mathbf{u} + \beta \mathbf{v} + \gamma \mathbf{w}. \quad (1.5)$$

A basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is positively oriented if 

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) > 0. \quad (1.6)$$

Two bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ have the same orientation if

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \text{ and } \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \text{ have the same sign.} \quad (1.7)$$

A basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal if

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0 \quad \text{and} \quad |\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1, \quad (1.8)$$

so that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually orthogonal and of unit length.

A standard method of showing that two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal uses the following result:

$$\mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v} \text{ for all vectors } \mathbf{v} \text{ if and only if } \mathbf{a} = \mathbf{b}. \quad (1.9)$$

The verification of this result is not difficult. Assume that $\mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$ for all $\mathbf{v}$. Then the choice $\mathbf{v} = \mathbf{a} - \mathbf{b}$ yields $|\mathbf{a} - \mathbf{b}|^2 = 0$ and, hence, $\mathbf{a} = \mathbf{b}$. Similarly,

$$\mathbf{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v} \text{ for all vectors } \mathbf{v} \text{ if and only if } \mathbf{a} = \mathbf{b}, \quad (1.10)$$

which is a result whose proof we leave as an exercise.

A subset $\mathcal{K}$ of vectors is referred to as a subspace if, given any vectors $\mathbf{u}$ and $\mathbf{v}$ belonging to $\mathcal{K}$ and any scalars $\alpha$ and $\beta$, the linear combination

$$\alpha \mathbf{u} + \beta \mathbf{v} \text{ belongs to } \mathcal{K}. \quad (1.11)$$

Examples of subspaces of $\mathcal{V}$ are the singleton $\{\mathbf{0}\}$, a line through the origin, a plane through the origin, and $\mathcal{V}$ itself. There are no other examples.

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7 We view a positively oriented basis as right-handed, since the definition of the cross product is based on the right-hand screw rule (Footnote 6).
EXERCISE

1. Verify (1.8).

1.2 Cartesian Coordinate Frames

Throughout this book, lowercase Latin subscripts range over the subset of integers \{1, 2, 3\}.

A Cartesian coordinate frame for \(\mathbb{E}\) consists of a reference point \(\mathbf{o}\) called the origin together with a positively oriented orthonormal basis \(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\) for \(\mathcal{V}\). Being positively oriented and orthonormal, the basis vectors obey

\[
\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad \text{and} \quad \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \epsilon_{ijk}.
\]

Here \(\delta_{ij}\), the Kronecker delta, is defined by

\[
\delta_{ij} = \begin{cases} 
1, & \text{if } i = j, \\
0, & \text{if } i \neq j,
\end{cases}
\]

while \(\epsilon_{ijk}\), the alternating symbol, is defined by

\[
\epsilon_{ijk} = \begin{cases} 
1, & \text{if } \{i, j, k\} = \{1, 2, 3\}, \{2, 3, 1\}, \text{or } \{3, 1, 2\}, \\
-1, & \text{if } \{i, j, k\} = \{2, 1, 3\}, \{1, 3, 2\}, \text{or } \{3, 2, 1\}, \\
0, & \text{if an index is repeated},
\end{cases}
\]

and, hence, has the value +1, −1, or 0 according to whether \(\{i, j, k\}\) is an even permutation, an odd permutation, or not a permutation of \(\{1, 2, 3\}\).

For brevity,

\[
\{\mathbf{e}_i\} \overset{\text{def}}{=} \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}
\]
denotes a positively oriented orthonormal basis.

1.3 Summation Convention. Components of a Vector and a Point

Throughout this book, we employ the Einstein summation convention according to which summation over the range 1, 2, 3 is implied for any index that is repeated twice in any term, so that, for instance,

\[
u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3,
\]

\[
S_{ij} u_j = S_{i1} u_1 + S_{i2} u_2 + S_{i3} u_3,
\]

\[
S_{ik} T_{kj} = S_{i1} T_{1j} + S_{i2} T_{2j} + S_{i3} T_{3j}.
\]

In the expression \(S_{ij} u_j\), the subscript \(i\) is free, because it is not summed over, while \(j\) is a dummy subscript, since

\[
S_{ij} u_j = S_{ik} u_k = S_{im} u_m.
\]

When an expression in which an index is repeated twice but summation is not to be performed we state so explicitly. For example,

\[
u_i v_i \quad \text{(no sum)}
\]

signifies that the subscript \(i\) is not to be summed over.
1.3 Summation Convention. Components of a Vector and a Point

Next, if $S_{jk} = S_{kj}$, then

$$
\epsilon_{ijk} S_{jk} = \epsilon_{ikj} S_{kj} = -\epsilon_{ijk} S_{jk} = -\epsilon_{ikj} S_{kj}
$$

and vice versa; therefore

$$
S_{ij} = S_{ji} \quad \text{if and only if} \quad \epsilon_{ijk} S_{jk} = 0. \quad (1.13)
$$

Because $\{e_i\}$ is a basis, every vector $u$ admits the unique expansion

$$
u = u_i e_i; \quad (1.14)
$$

the scalars $u_i$ are called the (Cartesian) components of $u$ (relative to this basis). If we take the inner product of (1.14) with $e_i$, we find that, since $e_i \cdot e_j = \delta_{ij}$,

$$
u_i = u_i \cdot e_i. \quad (1.15)
$$

Guided by this relation, we define the coordinates of a point $x$ with respect to the origin $o$ by

$$
x_i = (x - o) \cdot e_i. \quad (1.16)
$$

In view of (1.14), the inner and cross products of vectors $u$ and $v$ may be expressed as

$$
u \cdot v = (u_i e_i) \cdot (v_j e_j) = u_i v_j \delta_{ij} = u_i v_i \quad (1.17)
$$

and

$$
u \times v = (u_j e_j) \times (v_k e_k) = \epsilon_{ijk} u_j v_k e_i. \quad (1.18)
$$

In particular, (1.18) implies that the vector $u \times v$ has the component form

$$
(u \times v)_i = \epsilon_{ijk} u_j v_k. \quad (1.19)
$$

When working with the cross product, the epsilon-delta identity

$$
\epsilon_{ijk} \epsilon_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}, \quad (1.20)
$$

and its consequences

$$
\epsilon_{ijk} \epsilon_{ijl} = 2 \delta_{kl} \quad \text{and} \quad \epsilon_{ijk} \epsilon_{ijk} = 6, \quad (1.21)
$$

are useful. Also useful is the identity

$$
e_i = \frac{1}{2} \epsilon_{ijk} e_j \times e_k. \quad (1.22)
$$
Let \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) be vectors. Useful relations involving the inner and cross products then include

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}),
\]
\[
\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0,
\]
\[
\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u},
\]

EXERCISES

1. Verify (1.22) and (1.23).
2. Establish the following identities for any vectors \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \):

\[
(u \cdot v)^2 + |\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2,
\]
\[
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0.
\]
2 Tensor Algebra

2.1 What Is a Tensor?

We use the term tensor as a synonym for the phrase “linear transformation from $V$ into $V$.” A tensor $S$ is therefore a linear mapping of vectors to vectors; that is, given a vector $u$,

$$v = Su$$

is also a vector. One might think of a tensor $S$ as a machine with an input and an output: if a vector $u$ is the input, then the vector $v = Su$ is the output (Figure 2.1). The linearity of a tensor $S$ is embodied by the requirements:

$$S(u + v) = Su + Sv$$
$$S(\alpha u) = \alpha Su$$

for all vectors $u$ and $v$; for all vectors $u$ and scalars $\alpha$.

Tensors $S$ and $T$ are equal if their outputs are the same whenever their inputs are equal; precisely,

$$S = T$$

if and only if $Sv = Tv$ for all vectors $v$.

One way of showing that tensors $S$ and $T$ are equal is a consequence of the following result:

$$a \cdot Sb = a \cdot Tb$$

for all vectors $a$ and $b$ if and only if $S = T$.

To prove this, we write $a \cdot Sb = a \cdot Tb$ in the form $a \cdot (Sb - Tb)$, which, by (1.7), holds for all $a$ if and only if $Sb = Tb$ and the validity of this for all $b$ yields, by (2.2), $S = T$.

Consistent with (2.2), tensors are generally defined by their actions on arbitrary vectors. For example, the sum $S + T$ of tensors $S$ and $T$ and the product $\alpha S$ of a tensor $S$ and a scalar $\alpha$ are defined as follows:

$$(S + T)v = Sv + Tv,$$

$$(\alpha S)v = \alpha (Sv),$$

for all vectors $v$. As a consequence of these definitions, the set of all tensors forms a vector space (of dimension 9).

EXERCISE

1. Show that $S + T$ and $\alpha S$ defined by (2.4) are actually linear and hence tensors.
2.2 Zero and Identity Tensors. Tensor Product of Two Vectors. Projection Tensor. Spherical Tensor

Two basic tensors are the zero tensor \( 0 \) and the identity tensor \( 1 \), defined by

\[
0v = 0 \quad \text{and} \quad 1v = v
\]

for all vectors \( v \).

Another example of a tensor is the tensor product \( u \otimes v \), of two vectors \( u \) and \( v \), defined by

\[
(u \otimes v)w = (v \cdot w)u
\]

for all \( w \). By (2.5), the tensor \( u \otimes v \) maps any vector \( w \) onto a scalar multiple of \( u \).

Let \( e \) be a unit vector. Then, since

\[
(e \otimes e)u = (u \cdot e)e
\]

for any vector \( u \), the tensor \( e \otimes e \) maps each vector \( u \) to the projection \( (u \cdot e)e \) of \( u \) onto the vector \( e \). Similarly,

\[
(1 - e \otimes e)u = u - (u \cdot e)e,
\]

so that, for any vector \( u \), the tensor \( 1 - e \otimes e \) maps each vector \( u \) to the projection \( u - (u \cdot e)e \) of \( u \) onto the plane perpendicular to \( e \). The tensors

\[
e \otimes e \quad \text{and} \quad 1 - e \otimes e
\]

therefore define projections onto \( e \) and onto the plane perpendicular to \( e \) (Figure 2.2).

Next, note that

\[
1v = v
\]

\[
= (v \cdot e_i)e_i
\]

\[
= (e_i \otimes e_i)v
\]

for all \( v \); thus, by (2.2), we have the useful identity

\[
1 = e_i \otimes e_i. \quad (2.7)
\]

Finally, a tensor \( S \) of the form

\[
S = \alpha 1,
\]

with \( \alpha \) a scalar, is called a spherical tensor.
2.3 Components of a Tensor

Figure 2.2. The projections \((e \otimes e)u = (u \cdot e)e\) and \((1 - e \otimes e)u = u - (u \cdot e)e\) of vector \(u\) onto a unit vector \(e\) and onto the plane perpendicular to that unit vector.

EXERCISE

1. Show that, for \(a\) and \(b\) nonzero,

\[
a \otimes b = c \otimes d \quad \text{if and only if} \quad c = \alpha a \quad \text{and} \quad d = \beta b \quad \text{with} \quad \alpha \beta = 1.
\]

2.3 Components of a Tensor

Given a tensor \(S\), choose an arbitrary vector \(u\) and let

\[v = Su.\]

Then,

\[v_i e_i = S(u_j e_j) = u_j S e_j\]

and, by (1.10), taking the inner product of this relation with \(e_i\) yields

\[v_i = (e_i \cdot S e_j)u_j;\]

thus, defining the components of \(S\) by

\[S_{ij} = (S)_{ij} = e_i \cdot S e_j, \tag{2.9}\]

we see that the component form of the relation \(v = Su\) is

\[v_i = S_{ij} u_j. \tag{2.10}\]

Further, this relation implies that

\[v = v_i e_i = (S_{ij} u_j) e_i = (S_{ij} e_j \cdot u) e_i = (S_{ij} e_i \otimes e_j) u,\]

and since \(v = Su\) we must have

\[Su = (S_{ij} e_i \otimes e_j) u.\]